The U-plane of rank-one 4d $\mathcal{N} = 2$ KK theories

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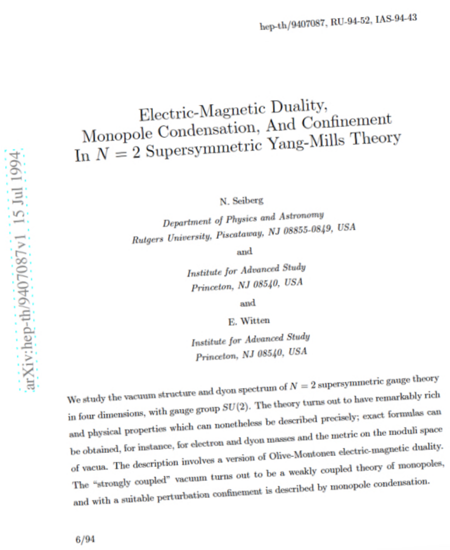
Preamble: the SW solution for SQCD

The Seiberg-Witten solution

Let us first go back to 'basics': [Seiberg, Witten, 1994]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Electric-Magnetic Duality,
Monopole Condensation, And Confinement
In $N = 2$ Supersymmetric Yang-Mills Theory

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We study the vacuum structure and dyon spectrum of $N = 2$ supersymmetric gauge theory in four dimensions, with gauge group $SU(2)$. The theory turns out to have remarkably rich and physical properties which can nonetheless be described precisely; exact formulas can be obtained, for instance, for electron and dyon masses and the metric on the moduli space of vacua. The description involves a version of Olive-Montonen electric-magnetic duality. The "strongly coupled" vacuum turns out to be a weakly coupled theory of monopoles, and with a suitable perturbation confinement is described by monopole condensation.

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Monopoles, Duality and Chiral Symmetry Breaking
in N=2 Supersymmetric QCD

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We study four dimensional $N = 2$ supersymmetric gauge theories with matter multiplets. r all such models for which the gauge group is $SU(2)$, we derive the exact metric on moduli space of quantum vacua and the exact spectrum of the stable massive states. mber of new physical phenomena occur, such as chiral symmetry breaking that is by the condensation of magnetic monopoles that carry global quantum numbers. e cases in which conformal invariance is broken only by mass terms, the formal- natically gives results that are invariant under electric-magnetic duality. In one his duality is mixed in an interesting way with $SO(8)$ triality.

4d $\mathcal{N} = 2$ SQCD

We are interested in 4d $\mathcal{N} = 2$ supersymmetric gauge theories. For simplicity, focus on SQCD with $SU(2)$ gauge group:

- Vector multiplet for gauge group $SU(2)$:

  $$\mathcal{V} = (\phi, A_\mu, \lambda_I, \bar{\lambda}^I, D_{IJ})$$

  Scalar potential includes term $V = \left|[\bar{\phi}, \phi]\right|^2 \geq 0$.
- N_f 'flavors': hypermultiplets in the fundamental, $\mathbf{2} \oplus \bar{\mathbf{2}}$, with masses m_i.
- Flavour symmetry algebra $\mathfrak{g}_F$: $\mathfrak{so}(2N_f)$ if $m_i = 0$, $\forall i$, $\mathfrak{u}(N_f)$ if $m_i = m$, and $\mathfrak{u}(1)^{N_f}$ with generic masses.
- Asymptotic freedom implies $N_f \leq 4$. The theory with $N_f = 4$ and $\mathfrak{g}_F = \mathfrak{so}(8)$ is a 4d SCFT with an exactly marginal gauge coupling.

4d $\mathcal{N} = 2$ SQCD

- Generic vacuum is on the **Coulomb branch:**

$$\phi = -\frac{i}{\sqrt{2}} \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} , \qquad SU(2) \to U(1)$$

  The SW solution gives the exact low-energy effective action for the IR $U(1)$:

$$S = \int d^4x \, \mathrm{Im}\big(\tau(a)\big) \left(F_{\mu\nu}F^{\mu\mu} + \partial_\mu a \partial^\mu a + \cdots\right)$$

- By supersymmetry, the CB metric is determined by an holomorphic function, the prepotential:

$$\tau = \frac{\partial^2 \mathcal{F}}{\partial a^2}$$

- The CB is parameterised by the gauge-invariant parameter:

$$u = \langle \mathrm{Tr}(\phi^2) \rangle \approx -a^2 + \cdots$$

  The CB of 4d $\mathcal{N} = 2$ SQCD is 'the u-plane'.

  The point at infinity, $u = \infty$, is the weak coupling point.

The u-plane of SQCD

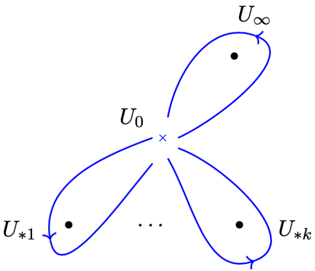
Electric-magnetic duality of a $U(1)$ vector multiplet:

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \to \mathbb{M}_* \begin{pmatrix} a_D \\ a \end{pmatrix} \;, \quad \mathbb{M}_* \in \mathrm{SL}(2,\mathbb{Z}) \cong \left\langle S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \;,\; T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle$$

with $SL(2,\mathbb{Z})$ monodromies of the 'electromagnetic periods' (modulo constant shifts if $m_i \neq 0$). We have:

$$a_D = \frac{\partial \mathcal{F}}{\partial a} \;, \qquad \tau = \frac{\partial a_D}{\partial a} \;.$$

For fixed masses, the u-plane has the form:

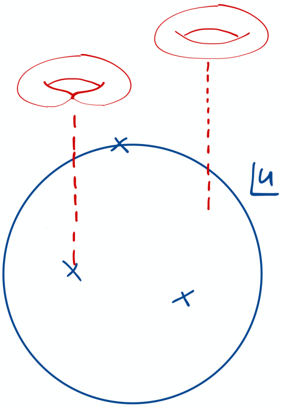


- paths $\boldsymbol{\gamma}_v$, $v = 1, \cdots, k$, and $v = \infty$.
- $\boldsymbol{\gamma}_\infty = -(\boldsymbol{\gamma}_1 + \cdots + \boldsymbol{\gamma}_k)$
- If m_i generic, $k = N_f + 2$.
- $\mathbb{M}_\infty \prod_{l=1}^{k} \mathbb{M}_{*l} = \mathbf{1}$.

We will think of the u-plane as a projective plane, $\mathbb{P}^1 \cong \{u\}$ with a distinguished point $u = \infty$.

The SW solution

Postulate that τ with $\mathrm{Im}(\tau) \geq 0$ is the modular parameter of an elliptic curve, E_u:



- We then have:

$$\tau = \frac{\omega_D}{\omega_a} = \frac{\partial a_D}{\partial a} \,,$$
$$\omega_D = \frac{da_D}{du} = \int_{\gamma_B} \boldsymbol{\omega} \,,$$
$$\omega_a = \frac{da}{du} = \int_{\gamma_A} \boldsymbol{\omega} \,.$$

- The SW solution is a specific **elliptic fibration over the CB.** The one-parameter family of curves E_u is usually called 'the SW curve'.
- The 'Seiberg-Witten geometry' is the total space of the SW fibration over the u-plane.
- It necessarily has singular fibers.
  Kodaira classification.

The SW solution

- **Singularity at infinity** determined at weak coupling (1-loop β-function):

$$I^*_{4-N_f} \quad : \qquad \mathbb{M}_\infty = -T^{4-N_f}$$

- **Simple singularities in the interior:** I_n singularity (multiplicative fiber):

$$I_n \quad : \qquad \mathbb{M}_* = T^n$$

  The actual monodromy is conjugate to T^n.
  If a single dyon of charge (m, q) becomes massless at $u = u_*$:

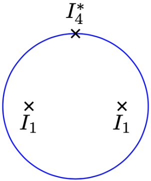
$$\mathbb{M}_*^{(m,q)} = B^{-1} T B = \begin{pmatrix} 1+mq & q^2 \\ -m^2 & 1-mq \end{pmatrix} .$$

- **Other possibilities,** from the Kodaira classification of singular elliptic fibers:

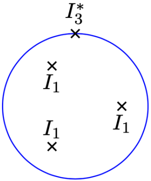
$$II \;:\quad \mathbb{M}_* = (ST)^{-1} , \qquad II^* \;:\quad \mathbb{M}_* = ST ,$$
$$III \;:\quad \mathbb{M}_* = S^{-1} , \qquad III^* \;:\quad \mathbb{M}_* = S ,$$
$$IV \;:\quad \mathbb{M}_* = (ST)^{-2} , \qquad IV^* \;:\quad \mathbb{M}_* = (ST)^2 .$$

The u-plane of massless SQCD

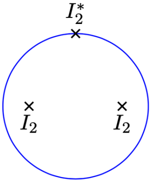
For massless SQCD, we have: [Seiberg, Witten, 1994]



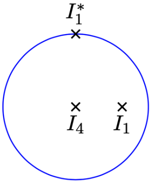
(a) $N_f = 0$.



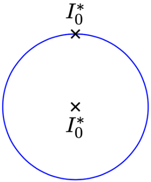
(b) $N_f = 1$.



(c) $N_f = 2$.



(d) $N_f = 3$.



(e) $N_f = 4$.

I_n singularity: n mutually local particles become massless.

The symmetry group of 4d $\mathcal{N}=2$ SQCD

The (global) symmetry group of a theory is, by definition, the group that acts effectively on gauge-invariant states. In particular, we must quotient by gauge redundancies.

The global symmetry of **massless** SQCD is easily determined **in the UV:**

$$G_F = SO(2N_f)/\mathbb{Z}_2$$

We also write this as:

N_f	0	1	2	3	4
G_F	-	$U(1)$	$(SU(2)/\mathbb{Z}_2)\times(SU(2)/\mathbb{Z}_2)$	$SU(4)/\mathbb{Z}_4$	$\mathrm{Spin}(8)/(\mathbb{Z}_2\times\mathbb{Z}_2)$

The pure $SU(2)$ gauge theory $(N_f=0)$ has a one-form symmetry:

[Gaiotto, Kapustin, Seiberg, Willett, 2014]

$$\mathcal{Z}^{[1]} = \mathbb{Z}_2$$

which acts on Wilson loops in the fundamental (*i.e.* background quark worldlines):

$$\mathbb{Z}_2 \ : \ W \to -W$$

The symmetry group of 4d $\mathcal{N} = 2$ SQCD

We would like to determine the symmetry directly **in the IR**.

Let us start with a partial answer:

Claim: The semi-simple part of the flavor symmetry algebra $\mathfrak{g}_F^{\rm NA} = {\rm Lie}(G_F)^{\rm NA}$ is given in terms of the Kodaira singularities in the interior:

$$\mathfrak{g}_F^{\rm NA} = \bigoplus_{v=1}^{k} \mathfrak{g}_v \ ,$$

with:

F_v	I_n	I_m^*	II	III	IV	II^*	III^*	IV^*
$\mathfrak{g}_v$	$\mathfrak{su}(n)$	$\mathfrak{so}(8+2m)$	$-$	$\mathfrak{su}(2)$	$\mathfrak{su}(3)$	$\mathfrak{e}_8$	$\mathfrak{e}_7$	$\mathfrak{e}_6$

We will soon explain how to determine G_F itself, directly from the SW geometry.

Rational elliptic surfaces and rational sections

SW curve and periods: generalities

It is convenient to bring the SW curve into the Weierstrass normal form:

$$y^2 = 4x^3 - g_2(u,m)x - g_3(u,m)$$

The singular fibers are located along the zeros of the discriminant:

$$\Delta(u) = g_2(u)^3 - 27g_3(u)^2$$

For SQCD, this is a polynomial of order $N_f + 2$. At generic masses, we have $N_f + 2$ simple roots in u (giving rise to I_1 singularities).

Example: For pure $SU(2)$, we have:

$$g_2(u) = \frac{4u^2}{3} - 4\Lambda^4 \ , \qquad g_3(u) = -\frac{8u^3}{27} + \frac{4}{3}u\Lambda^4 \ ,$$

and the discriminant:

$$\Delta = 16\Lambda^8 \left(u^2 - 4\Lambda^4\right)$$

SW curve and periods: generalities

Kodaira's classification of singularities of elliptic fibrations:

$$g_2 \sim (u-u_*)^{\mathsf{ord}(g_2)} \,, \qquad g_3 \sim (u-u_*)^{\mathsf{ord}(g_3)} \,, \qquad \Delta \sim (u-u_*)^{\mathsf{ord}(\Delta)} \,.$$

fiber	τ	$\mathrm{ord}(g_2)$	$\mathrm{ord}(g_3)$	$\mathrm{ord}(\Delta)$	M_*	flavor
I_k	$i\infty$	0	0	k	T^k	$\mathfrak{su}(k)$
I_k^*	$i\infty$	2	3	$k+6$	$-T^k$	$\mathfrak{so}(2k+8)$
I_0^*	τ_0	≥ 2	≥ 3	6	$-\mathbf{1}$	$\mathfrak{so}(8)$
II	$e^{\frac{2\pi i}{3}}$	≥ 1	1	2	$(ST)^{-1}$	-
II^*	$e^{\frac{2\pi i}{3}}$	≥ 4	5	10	(ST)	$\mathfrak{e}_8$
III	i	1	≥ 2	3	S^{-1}	$\mathfrak{su}(2)$
III^*	i	3	≥ 5	9	S	$\mathfrak{e}_7$
IV	$e^{\frac{2\pi i}{3}}$	≥ 2	2	4	$(ST)^{-2}$	$\mathfrak{su}(3)$
IV^*	$e^{\frac{2\pi i}{3}}$	≥ 3	4	8	$(ST)^2$	$\mathfrak{e}_6$

SW curve and periods: generalities

We are interested in the ‘physical periods’:

$$a_D = \int_{\gamma_B} \lambda_{\rm SW} \;, \qquad\qquad a = \int_{\gamma_A} \lambda_{\rm SW} \;.$$

with the Seiberg-Witten differential such that:

$$\frac{d\lambda_{\rm SW}}{du} = \boldsymbol{\omega} \;, \qquad\qquad \boldsymbol{\omega} \equiv \frac{dy}{x}$$

Thus, we can find the physical periods from the ‘geometric periods’:

$$\omega_D = \int_{\gamma_B} \boldsymbol{\omega} \;, \qquad\qquad \omega_a = \int_{\gamma_A} \boldsymbol{\omega} \;.$$

At any fixed m, they satisfy a standard Picard-Fuchs equation:

$$\Delta(u) \frac{d^2\omega}{du^2} + P(u) \frac{d\omega}{du} + Q(u)\, \omega = 0$$

SW geometry and rational elliptic surface

The low-energy physics on the CB is determined by the (affine) bundle:

$$\mathbb{C}^2 \to \text{(SW geom)} \to \overline{\mathcal{B}} \cong \{u\}$$

with the fibers given by the periods (a_D, a).

Once we geometrize the periods by introducing the SW curve E_u, we have:

$$E \to \mathcal{S} \to \overline{\mathcal{B}}$$

We compactify the base by adding the point at infinity:

$$\overline{\mathcal{B}} \cong \{u\} \cong \mathbb{P}^1$$

The SW geometry $\mathcal{S}$ is then a **rational elliptic surface (RES) with a section.**

Note: Any (resolved) RES $\tilde{\mathcal{S}}$ can be obtained as a blow up of the projective plane at 9 points, $dP_9 = \mathrm{Bl}_9(\mathbb{P}^2)$. This is also called 'half-K3 surface' by string theorists. A deep fact is then that:

$$H_2(\tilde{\mathcal{S}}, \mathbb{Z}) \cong \langle (O), E \rangle \oplus (-E_8)$$

with E_8 denoting the E_8 lattice, for the 2-cycles with the intersection pairing.

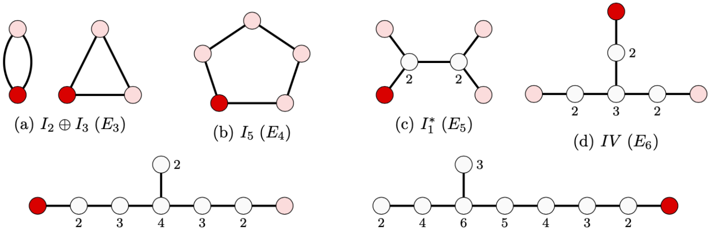
SW geometry and rational elliptic surface

The singular fibers lead to ADE singularities on $\mathcal{S}$, in correspondence with the ADE 'flavor' type.

They admit a standard resolution, $\tilde{\mathcal{S}} \to \mathcal{S}$. (Kodaira-Neron model.)

$$\pi^{-1}(U_{*,v}) = F_v \cong \sum_{i=0}^{m_v - 1} \widehat{m}_{v,i} \Theta_{v,i} \ ,$$

Example: The E_n family.



(a) $I_2 \oplus I_3$ (E_3) (b) I_5 (E_4) (c) I_1^* (E_5) (d) IV (E_6)

(e) III (E_7) (f) II (E_8)

The Mordell-Weil group of rational section

Elliptic curves are additive groups:

$$P_1 + P_2 = P_3$$

Given an elliptic fibration $E \to \mathcal{S} \to \mathbb{P}^1$, there may exist non-trivial rational sections. In Weierstrass form:

$$P = (x(u), y(u)) \ , \qquad x(u),\ y(u) \in \mathbb{C}(u)$$

They form a finitely generated abelian group, the Mordell-Weil group:

$$\Phi = \mathrm{MW}(\mathcal{S}) \cong \mathbb{Z}^{\mathrm{rk}(\Phi)} \oplus \mathbb{Z}_{k_1} \oplus \cdots \oplus \mathbb{Z}_{k_t} \ .$$

The number of free generators, $\mathrm{rk}(\Phi) \geq 0$, is called the rank of the MW group.

The trivial element in Φ is the zero section, $O = (\infty, \infty)$.

Importantly, the MW group can have non-trivial torsion elements, $k_i P_{\mathrm{tor}} = O$.

The classification of rational elliptic surfaces

Rational elliptic surfaces $\mathcal{S}$ are fully classified. [Persson, 1990; Miranda, 1990]
They are characterised by:

- A set of 'allowed' singular fibers, (F_v).
- The MW group Φ.

In fact, in most cases, the set of singular fibers fully determines $\mathcal{S}$.

A basic but powerful global constraint is:

$$\sum_v \mathrm{ord}(\Delta)|_{U_{*v}} = 12$$

where the sum includes '$v = \infty$'. There is thus a finite set of allowed singularities. Additional considerations show that these are the following 20:

$$I_1, \cdots, I_9 \ , \quad I_0^*, \cdots, I_4^* \ , \quad , II \ , \ III \ , \ IV \ , \ II^* \ , \ III^* \ , \ IV^* \ .$$

Total number of distinct RES: **289**.

4d SQFTs of rank one, revisited

Fixing the fiber at infinity

The RES perspective, and Persson's classification, gives us a bird's-eye view of rank-one 4d $\mathcal{N}=2$ theories.

The basic idea, generalising [Caorsi, Cecotti, 2018], is that the UV $\mathcal{N}=2$ SQFT is determined by the fiber at infinity:

$$\mathcal{T}_{F_\infty} \qquad \longleftrightarrow \qquad \{\mathcal{S} \mid \pi^{-1}(\infty) = F_\infty\} \ .$$

F_∞	I_1	I_2	I_3	I_4	I_5	I_6	I_7	I_8	I_9
$D_{S^1}\mathcal{T}_{5\text{d}}$	E_8	E_7	E_6	E_5	E_4	E_3	E_2	E_1 or $\widetilde{E}_1$	E_0
$\#\mathcal{S}$	227	140	77	51	26	16	6	$2+2$	1
F_∞	II	III	IV	I_0^*	I_1^*	I_2^*	I_3^*	I_4^*	
$\mathcal{T}_{4\text{d}}$	E_8	E_7	E_6	D_4	D_3	D_2	$N_f=1$	$N_f=0$	
$\#\mathcal{S}$	137	93	49	19	13	6	2	1	
F_∞					IV^*	III^*	II^*		
$\mathcal{T}_{4\text{d}}$					$A_2\ (H_2)$	$A_1\ (H_1)$	$-\ (H_0)$		
$\#\mathcal{S}$					8	4	2		

MN theories

AD theories

SQCD

Fixing the fiber at infinity

Some comments:

- Fixing F_∞, the list of distinct RES with such a fiber gives the number of distinct CB configurations for $\mathcal{T}_{F_\infty}$, which we denote by:

$$\mathcal{S} \cong (F_\infty, F_1, \cdots, F_k)$$

  For instance, pure $SU(2)$ has a single CB configuration, $\mathcal{S} \cong (I_4^*, I_1, I_1)$.
- The above 'periodic table' includes the 3 'classic AD SCFTs [Argyres, Douglas, 1995] and the 3 E_n MN theories [Minahan, Nemeschansky, 1996].
- It does not include the other 4d SCFTs [Argyres, Wittig, 2007; Argyres, Lotito, Lu, Martone, 2016] with enhanced CB (although, see [Caorsi, Cecotti, 2016]).
- *Conjecture (?):* the table gives the full list of CB configurations for rank-one 4d $\mathcal{N} = 2$ SQFTs with a 'trivial' CB (*i.e.* with only a $U(1)$ vector multiplet).
- The top row corresponds to 5d SCFTs on $\mathbb{R}^4 \times S^1$, as we will show.
- If we choose $F_\infty = I_0$ (the trivial fiber), we get the E-string on $\mathbb{R}^4 \times T^2$. There are therefore 289 distinct CB configurations for that theory.

Symmetry group and rational sections

We claimed above that the non-abelian part of the flavour symmetry was captured by the singular fibers (in the interior), $F_{v \neq \infty}$.

We also claim that each generator of $\Phi_{\rm free} = \Phi/\Phi_{\rm tor}$ gives rise to a $U(1)$ flavor symmetry.

The full flavour symmetry algebra is then:

$$\mathfrak{g}_F = \bigoplus_{s=1}^{\mathrm{rk}(\Phi)} \mathfrak{u}(1)_s \oplus \bigoplus_{v=1}^{k} \mathfrak{g}_v \ ,$$

One can also show that:

$$\mathrm{rank}(\mathfrak{g}_F) = 8 - \mathrm{rank}(\mathfrak{g}_\infty) \ .$$

Example: $SU(2)$, $N_f = 1$. The massless CB configuration is $\mathcal{S} \cong (I_3^*, 3I_1)$. In that case, one indeed finds $\Phi \cong \mathbb{Z}$, in agreement with $\mathfrak{g}_F = \mathfrak{u}(1)$.

Symmetry group and rational sections

The **global form of flavour group** can be determined by analysing the full MW group. For simplicity, assume that $\mathrm{rk}(\Phi) = 0$, so that G_F is semi-simple:

$$\Phi = \Phi_{\mathrm{tor}} = \mathbb{Z}_{k_1} \oplus \cdots \oplus \mathbb{Z}_{k_t}$$

Let $\tilde{G}_F$ denote the simply-connected group such that $\mathfrak{g}_F = \mathrm{Lie}(G_F)$.

Define the subgroup of Φ_{tor} of 'interior-narrow sections':

$$\mathcal{Z}^{[1]} = \left\{ P \in \Phi_{\mathrm{tor}} \;\middle|\; (P) \text{ intersects } \Theta_{v,0} \text{ for all } F_{v\neq\infty} \right\} ,$$

and denote by $\mathscr{F}$ the cokernel of the inclusion map $\mathcal{Z}^{[1]} \to \Phi_{\mathrm{tor}}$:

$$0 \to \mathcal{Z}^{[1]} \to \Phi_{\mathrm{tor}} \to \mathscr{F} \to 0 \ .$$

Then, we claim that:

- $G_F = \tilde{G}_F / \mathscr{F}$ is the flavour symmetry group.
- $\mathcal{Z}^{[1]}$ is the one-form symmetry group.

Symmetry group and rational sections

The proof of the above statements goes through local mirror symmetry and borrows arguments from the F-theory literature. [Aspinwall, 1998; Mayrhofer, Morrison, Till, Weigand, 2014; Cvetic, Lin, 2017; Monnier, Moore, Park, 2017]. We will not go through it today.

A complementary way to understand the result is by taking the CB configuration of $\mathcal{T}_{F_\infty}$ with generic masses, so that we have the explicit symmetry breaking pattern:

$$G_F \to U(1)^{\mathrm{rank}(G_F)}$$

These $U(1)$'s are generated by sections in Φ_{free}. Furthermore, there is a natural lattice, the (narrow) Mordell-Weil lattice of $\mathcal{S}$, which was computed for any $\mathcal{S}$. [Shioda, 1990] Using these mathematical results, we can confirm the above claims in a case-by-case basis. (The narrow MWL is the weight lattice of G_F.)

Symmetry group and rational sections

Example: SQCD. For massless SQCD, one finds:

N_f	0	1	2	3	4
$\mathcal{S}$	$(I_4^*, 2I_1)$	$(I_3^*, 3I_1)$	$(I_2^*, 2I_2)$	(I_1^*, I_4, I_1)	(I_0^*, I_0^*)
Φ	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_4$	$\mathbb{Z}_2^2$

This matches the results expected from the UV:

- $N_f = 0$: we have $\Phi_{\rm tor} = \mathcal{Z}^{[1]} = \mathbb{Z}_2$, in agreement with known results.
- $N_f = 2$: we have $\Phi_{\rm tor} = \mathscr{F}$ and $G_F = SU(2) \times SU(2))/(\mathbb{Z}_2 \times \mathbb{Z}_2)$.
- $N_f = 3$: we have $\Phi_{\rm tor} = \mathscr{F}$ and $G_F = SU(4)/\mathbb{Z}_4$.
- $N_f = 4$: we have $\Phi_{\rm tor} = \mathscr{F}$ and $G_F = \mathrm{Spin}(8)/(\mathbb{Z}_2 \times \mathbb{Z}_2)$.

Symmetry group and rational sections

The general result can also be applied to non-Lagrangian theories. We have the following interesting RES: [Miranda, Persson, 1986]

- $\mathcal{S} = (II, II^*)$, with $\Phi = 0$.
  - If $F_\infty = II^*$, we have the AD point H_0 with trivial flavour group.
  - If $F_\infty = II$, we have the E_8 MN SCFT, with $G_F = E_8$.
- $\mathcal{S} = (III, III^*)$, with $\Phi = \mathbb{Z}_2$.
  - If $F_\infty = III^*$, we have the AD point H_1 with flavour group $G_F = SO(3)$.
  - If $F_\infty = III$, we have the E_7 MN SCFT, with $G_F = \mathrm{E}_7/\mathbb{Z}_2$.
- $\mathcal{S} = (IV, IV^*)$, with $\Phi = \mathbb{Z}_3$.
  - If $F_\infty = IV^*$, we have the AD point H_2 with flavour group $G_F = PSU(3)$.
  - If $F_\infty = IV$, we have the E_6 MN SCFT, with $G_F = \mathrm{E}_6/\mathbb{Z}_3$.

All these flavour groups are centerless. For the MN theories, this determination reproduces recent results [Bhardwaj, 2021]. The H_1 flavour group was determined in [Buican, Jiang, 2021], and the H_2 flavour group is a new result.

Systematic analysis of CB configurations

Using the Persson classification and some direct computations, we can map out the full set of CB configurations of a given SQFT $\mathcal{T}_\infty$, in principle.

Example: $SU(2)$, $N_f = 3$. There are 13 allowed configurations:

	$\{F_v\}$	m_1	m_2	m_3	$\mathfrak{g}_F$	$\mathrm{rk}(\Phi)$	Φ_{tor}
massless	I_1^*, I_4, I_1	0	0	0	A_3	0	$\mathbb{Z}_4$
u(3) sym.	$I_1^*, I_3, 2I_1$	m_1	m_1	m_1	$A_2 \oplus \mathfrak{u}(1)$	1	$-$
AD point H_2	I_1^*, IV, I_1	$\Lambda/2$	m_1	m_1	$A_2 \oplus \mathfrak{u}(1)$	1	$-$
	I_1^*, I_3, II	$-\Lambda/16$	m_1	m_1	$A_2 \oplus \mathfrak{u}(1)$	1	$-$
	I_1^*, III, I_2	$\Lambda/4$	0	0	$2A_1 \oplus \mathfrak{u}(1)$	1	$\mathbb{Z}_2$
	$I_1^*, 2I_2, I_1$	m_1	0	0	$2A_1 \oplus \mathfrak{u}(1)$	1	$\mathbb{Z}_2$
	I_1^*, III, II	$-\frac{7}{4}\Lambda$	$i\sqrt{2}\Lambda$	m_1	$A_1 \oplus 2\mathfrak{u}(1)$	2	$-$
	$I_1^*, III, 2I_1$	$\frac{m_2^2}{\Lambda} + \frac{\Lambda}{4}$	m_2	m_1	$A_1 \oplus 2\mathfrak{u}(1)$	2	$-$
	I_1^*, II, I_2, I_1	m_1	$\frac{(4m_1+\Lambda)^{3/2}}{6\sqrt{3\Lambda}}$	m_1	$A_1 \oplus 2\mathfrak{u}(1)$	2	$-$
	$I_1^*, I_2, 3I_1$	m_1	m_2	m_1	$A_1 \oplus 2\mathfrak{u}(1)$	2	$-$
	$I_1^*, 2II, I_1$	$\left(-2T_2\Lambda + \frac{13}{8}\Lambda^3,\ 5T_2\Lambda^2 - \frac{57}{16}\Lambda^4\right)$			$3\mathfrak{u}(1)$	3	$-$
	$I_1^*, II, 3I_1$	$\left(\frac{1}{4}T_2\Lambda - \frac{1}{16}\Lambda^3,\ \frac{1}{2}T_2\Lambda^2 - \frac{3}{16}\Lambda^4\right)$			$3\mathfrak{u}(1)$	3	$-$
generic	$I_1^*, 5I_1$	m_1	m_2	m_3	$3\mathfrak{u}(1)$	3	$-$

Modularity of the u-plane

For any 4d $\mathcal{N}=2$ SQFT with mass parameters m, we have an 'extended CB' where m are viewed as VEVs for background vector multiplets.

There are many 'special loci' on the extended Coulomb branch which have modular properties. More precisely, it can happen that, at some fixed values of the masses, **the u-plane is a modular curve:**

$$\overline{\mathcal{B}} \cong \mathbb{H}/\Gamma \,, \qquad\qquad \Gamma \subset SL(2,\mathbb{Z})$$

for some particular modular subgroup Γ. When this happens, the map:

$$u \;:\; \mathbb{H}/\Gamma \to \overline{\mathcal{B}} \;:\; \tau \mapsto u(\tau)$$

is an isomorphism. The Γ-invariant function $u(\tau)$ is called the Hauptmodul (or principal modular function) of Γ.

When the CB is modular, the singularities are in one-to-one correspondence with cusps and elliptic points of Γ. This simplifies the analysis of *e.g.* the monodromy group.

Note: even when the CB is not modular, it is advantageous to work on the τ-plane. See [Aspman, Furrer, Manschot, 2000, 2021] for recent discussions.

Modular curves for SQCD

Massless SQCD with $N_f \neq 1$ is modular: [Seiberg, Witten, 1994; Nahm, 1996]

Theory	$\Delta(u) = 0$	$F_{v\neq\infty}$	F_∞	*Modular Function*	*Monodromy*	*Cusps* τ
$N_f = 0$	$+1, -1$	I_1, I_1	I_4^*	$u(\tau) = 1 + \frac{1}{8}\left(\frac{\eta(\frac{\tau}{4})}{\eta(\tau)}\right)^8$	$\Gamma^0(4)$	$0, 2, i\infty$
$N_f = 1$	$u^3 = 1$	$3I_1$	I_3^*	$u^3 = \frac{2E_4(\tau)^{\frac{3}{2}}}{E_4(\tau)^{\frac{3}{2}} + E_6(\tau)}$	$\Gamma_{N_f=1}$	$0, 1, 2, i\infty$
$N_f = 2$	$+1, -1$	I_2, I_2	I_2^*	$u(\tau) = 1 + \frac{1}{8}\left(\frac{\eta(\frac{\tau}{2})}{\eta(2\tau)}\right)^8$	$\Gamma(2)$	$0, 1, i\infty$
$N_f = 3$	$0, 1$	I_4, I_1	I_1^*	$u(\tau) = -\frac{1}{16}\left(\frac{\eta(\tau)}{\eta(4\tau)}\right)^8$	$\Gamma_0(4)$	$0, -\frac{1}{2}, i\infty$

Note: Massless $N_f = 1$ is not modular.

Modular curves for SQCD

Example: pure $SU(2)$. Modular curve for $\Gamma^0(4)$. Two cusps of width 1.

$$u(\tau) = \frac{1}{8}\left(q^{-\frac{1}{4}} + 20q^{\frac{1}{4}} - 62q^{\frac{3}{4}} + 216q^{\frac{5}{4}} - 641q^{\frac{7}{4}} + 1636q^{\frac{9}{4}} + \mathcal{O}\left(q^{\frac{11}{4}}\right)\right) .$$

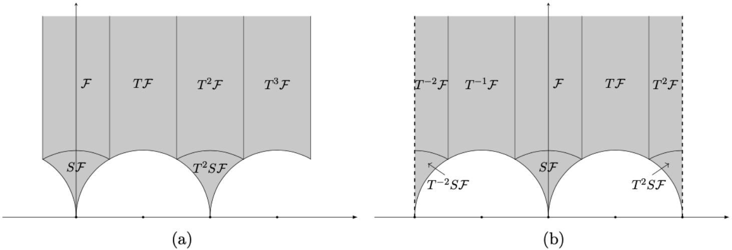


Figure 4: Fundamental domains for $\Gamma^0(4)$. Figure (a) shows a standard choice, with width one cusps at $\tau = 0$ and 2, while in figure (b) the cusp at $\tau = \pm 2$ is split, with the branch cut of the periods indicated by the dashed line.

Associated monodromies:

$$\mathbb{M}_{u=1} = STS^{-1} , \qquad \mathbb{M}_{u=-1} = (T^2S)T(T^2S)^{-1} , \qquad \mathbb{M}_{\infty} = PT^4 .$$

Modular curves for SQCD

Another example: $SU(2), N_f = 1$.

$\{F_v\}$	m_1	$\mathfrak{g}_F$	$\mathrm{rk}(\Phi)$	Φ_{tor}
$I_3^*, 3I_1$	m_1	$\mathfrak{u}(1)$	1	–
I_3^*, II, I_1	$m_1^3 = \frac{27}{16}\Lambda^3$	$\mathfrak{u}(1)$	1	–

Two configurations: massless one is not modular. The other is modular for $\Gamma = \Gamma^0(3)$:

$$u(\tau) = -\frac{5}{3} - \frac{1}{9}\left(\frac{\eta\left(\frac{\tau}{3}\right)}{\eta(\tau)}\right)^{12} ,$$

Note the AD points H_0 as an elliptic point: [Argyres, Plesser, Seiberg, Witten, 1995]

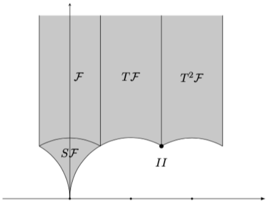


Figure 7: Fundamental domain for $\Gamma^0(3)$ corresponding to the configuration (I_3^*, I_1, II) on the CB of the 4d $SU(2)$, $N_f = 1$ theory. The marked point $\tau = 2 + e^{2i\pi/3}$ is the elliptic point of the congruence subgroup $\Gamma^0(3)$.

The U-plane of the E_n 5d SCFTs

Geometric engineering in IIA and M-theory

Consider a **Type IIA string theory on** $\mathbb{R}^4 \times \tilde{\mathbf{X}}$, with $\mathbf{X}$ a Calabi-Yau manifold. The low-energy theory is a 4d $\mathcal{N}=2$ supergravity theory. If $\mathbf{X}$ is non-compact, we have a 4d $\mathcal{N}=2$ QFT in the infrared. [Katz, Klemm, Vafa, 1996]

Plot twist: the low-energy QFT associated to $\tilde{\mathbf{X}}$ itself is 'secretly' five-dimensional. [Witten, 1995; Nekrasov, 1996]. Indeed, we may consider **M-theory on** $\mathbb{R}^5 \times \tilde{\mathbf{X}}$. If we take a smooth $\tilde{\mathbf{X}}$ which is a crepant resolution of a canonical singularity $\mathbf{X}$:

$$\tilde{\mathbf{X}} \to \mathbf{X}$$

we are on the Coulomb branch of a **5d SCFT** $\mathcal{T}_{\mathbf{X}}^{\rm 5d}$**.**

We then have:

$$\begin{aligned} \mathcal{T}_{\mathbf{X}}^{\rm 5d} \text{ on } \mathbb{R}^4 \times S^1 \quad &\leftrightarrow \quad \text{M-theory on } \mathbb{R}^4 \times S^1 \times \mathbf{X} \\ &\leftrightarrow \quad \text{IIA string theory on } \mathbb{R}^4 \times \mathbf{X} \end{aligned}$$

This gives us a **4d** $\mathcal{N}=2$ **supersymmetric** Kaluza-Klein (KK) field theory:

$$\boxed{D_{S^1}\mathcal{T}_{\mathbf{X}}^{\rm 5d} \text{ on } \mathbb{R}^4 \quad \cong \quad \mathcal{T}_{\mathbf{X}}^{\rm 5d} \text{ on } \mathbb{R}^4 \times S^1_\beta}$$

Geometric engineering in IIA and M-theory

Let us focus on the simplest example, of **rank one:** [Morrison, Seiberg, 1996]

$$\tilde{\mathbf{X}} = \mathrm{Tot}(\mathcal{K} \to S) \; , \qquad S = \mathbb{F}_0 \text{ or } dP_n \; (n \leq 8)$$

Singularity $\mathbf{X}$: blow-down the zero section S, which is a Fano surface.

Intersection form $H_2(S, \mathbb{Z}) \times H_2(S, \mathbb{Z}) \to \mathbb{Z}$ can be written as:

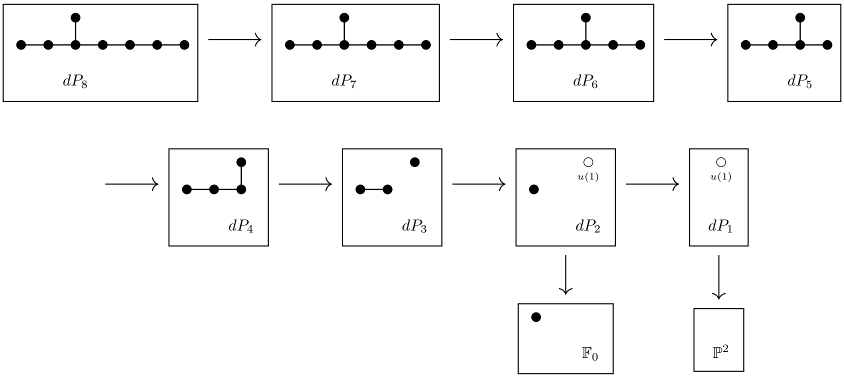
$$\begin{pmatrix} 9-n & 0 \\ 0 & -A^{E_n}_{IJ} \end{pmatrix} \; , \quad I, J = 1, \cdots, n \; , \qquad 9-n = \deg(S) = \mathcal{K} \cdot \mathcal{K}$$

$\Rightarrow$ M2-brane particles on CB form representations of $E_n = \mathfrak{e}_n$ algebra.

Note: We may also consider dP_9 (a RES.) The theory is then secretly six-dimensional. That is the E-string theory (a 6d $\mathcal{N} = (1,0)$ SCFT) on $\mathbb{R}^4 \times T^2$.

E_n theories from del Pezzos

These SCFTs are all related by RG flows triggered by massive deformations:



The 5d gauge theory limit

- These 10 rank-one SCFTs were first discovered by Seiberg as UV fixed points of 5d $\mathcal{N}=1$ gauge theories. [Seiberg, 1996]
- Recall that 5d gauge theories are IR-free effective theories. The perturbative gauge-theory description is valid for RG scales:

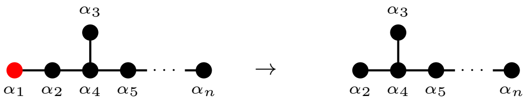
$$\mu \ll m_0 \equiv \frac{1}{g_{5\mathrm{d}}^2}$$

- $\mathcal{T}_{E_n}^{5\mathrm{d}}$ admits a mass deformation to a 5d $\mathcal{N}=1$ gauge theory in the IR:

$$E \ll m_0 = \frac{1}{g_{5\mathrm{d}}^2} \quad : \qquad \textbf{5d } \mathcal{N}=1\ SU(2) \textbf{ with } N_f = n-1 \textbf{ fundamentals.}$$

  This mass deformation breaks the flavor algebra as:

$$E_n \qquad \rightarrow \qquad \mathfrak{so}(2N_f) \oplus \mathfrak{u}(1)$$



The U-plane of $D_{S^1}\mathcal{T}_{\mathbf{X}}^{\mathrm{5d}}$

As a first approximation, we can then think of our E_n theories as 5d $SU(2)$ gauge theories. The low-energy $U(1)$ scalar is:

$$a = i(\varphi + iA_5) \; , \qquad\qquad e^{2\pi i A_5} \equiv e^{\int_{S^1} A}$$

and the gauge-invariant order parameter is:

$$\boxed{U = \langle W \rangle = e^{2\pi i a} + e^{-2\pi i a} + \cdots}$$

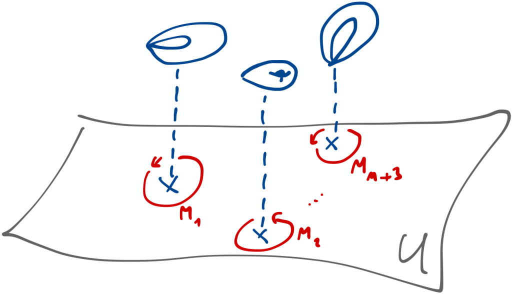
Here W is a supersymmetric Wilson line in 5d, wrapped along the S^1.

Similarly, the complexified mass parameters are flavor Wilson lines:

$$M_I = e^{2\pi i \mu_I} = e^{-\beta m_I + i\vartheta_I}$$

The U-plane of $D_{S^1}\mathcal{T}_{\mathbf{X}}^{5\mathrm{d}}$

At fixed M_I, the Coulomb branch is one-dimensional, with local coordinate $U \in \mathbb{C}$. This is **the U-plane.**



As in 'ordinary' 4d $\mathcal{N} = 2$ theories, the low-energy physics is fully determined by some Seiberg-Witten geometry. The E_n curves were derived in [Ganor, Morrison, Seiberg, 1996; Eguchi, Sakai, 2002].

The U-plane from local mirror symmetry

The SW solution is essentially local **mirror symmetry:** [Katz, Mayr, Vafa, 1996]

$$\text{CB of } D_{S^1}\mathcal{T}_{\mathbf{X}}^{\rm 5d} \quad \longleftrightarrow \quad \text{IIA string theory on } \mathbb{R}^4 \times \tilde{\mathbf{X}}$$
$$\longleftrightarrow \quad \text{IIB string theory on } \mathbb{R}^4 \times \hat{\mathbf{Y}}$$

We have the local mirror symmetry between smooth threefolds:

$$\tilde{\mathbf{X}} \quad \leftrightarrow \quad \hat{\mathbf{Y}}\,, \qquad\qquad D(\tilde{\mathbf{X}}) \quad \leftrightarrow \quad \text{Fuk}(\hat{\mathbf{Y}})$$

In particular:

- U, M_I are complex structure parameters of $\widehat{\mathbf{Y}}$.
- a, μ_I are Kähler parameters of $\tilde{\mathbf{X}}$.
- The exact expression:
$$a(U) = \frac{1}{2\pi i}\log\frac{1}{U} + \sum_k c_k U^k$$
is the mirror map.

The fiber at infinity

Consider the E_n theory. One can determine the large volume monodromy from the semi-classical periods.

Let us give a more "5d QFT" derivation: Take a limit where the 5d $SU(2)$, $N_f = n-1$ gauge-theory description is valid. At one-loop, the prepotential of the theory on $\mathbb{R}^4 \times S^1$ reads: [Nekrasov, 1998]

$$\mathcal{F} = \quad \mu_0 a^2 + \frac{2}{(2\pi i)^3}\mathrm{Li}_3\left(e^{4\pi i a}\right) - \frac{1}{(2\pi i)^3}\sum_{a=1}^{n-1}\sum_{\pm}\mathrm{Li}_3\left(e^{2\pi i(\pm a+\mu_a)}\right)$$

and $a_D = \frac{\partial \mathcal{F}}{\partial a}$. The large volume monodromy is:

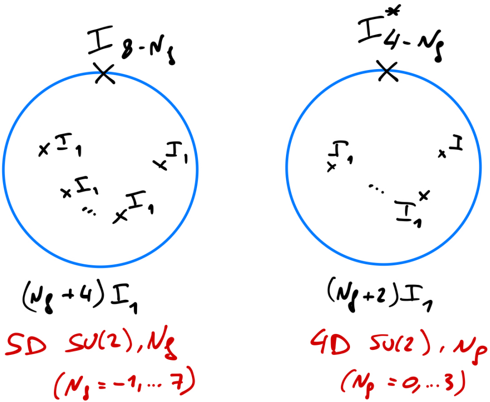
$$a_D \to a_D + (9-n)a + \mu_0 - \sum_{a=1}^{n-1}\mu_a\ , \qquad\qquad a \to a+1$$

We thus have:

$$\boxed{\mathbb{M}_\infty = T^{9-n} = \begin{pmatrix} 1 & 9-n \\ 0 & 1 \end{pmatrix}}$$

This determines **the fiber at infinity,** $F_\infty = I_{9-n}$**,** as anticipated.

Rational elliptic surfaces and *generic masses*:



The I_k fiber has monodromy conjugate to T^k. The bulk I_1 corresponds to a single BPS particle becoming massless:

$$M_*^{(m,q)} = B^{-1}TB = \begin{pmatrix} 1+mq & q^2 \\ -m^2 & 1-mq \end{pmatrix} .$$

The massless curves

Consider now $M_I = 1$. One finds:

$$
\begin{array}{lll}
E_8 & : & II^* \oplus I_1 & \Phi = 0 \\
E_7 & : & III^* \oplus I_1 & \Phi = \mathbb{Z}_2 \\
E_6 & : & IV^* \oplus I_1 & \Phi = \mathbb{Z}_3 \\
\hline
E_5 & : & I_1^* \oplus I_1 & \Phi = \mathbb{Z}_4 \\
E_4 & : & I_5 \oplus I_1 \oplus I_1 & \Phi = \mathbb{Z}_5 \\
E_3 & : & I_3 \oplus I_2 \oplus I_1 & \Phi = \mathbb{Z}_6 \\
E_2 & : & I_2 \oplus I_1 \oplus I_1 \oplus I_1 & \Phi = \mathbb{Z} \\
E_1 & : & I_2 \oplus I_1 \oplus I_1 & \Phi = \mathbb{Z}_2 \\
\tilde{E}_1 & : & I_1 \oplus I_1 \oplus I_1 \oplus I_1 & \Phi = \mathbb{Z} \\
E_0 & : & I_1 \oplus I_1 \oplus I_1 & \Phi = \mathbb{Z}_3
\end{array}
$$

in agreement with old 'classic' results. [Ganor, Morrison, Seiberg, 1996]

- This reproduce the E_n flavor symmetry, including abelian factors.
- The 4d LEEFT is IR free for $n < 6$

MW group and global symmetry

The general prescription for the global symmetry works here too. We find:

$$G_F = \mathrm{E}_n / Z(\mathrm{E}_n)$$

for the massless theories with semi-simple symmetry group.

- This agrees with the 5d result of [Apruzzi, Bhardwaj, Oh, Schafer-Nameki, 2021], which found G_F centerless using directly the M-theory geometry.
- The fiber $F_\infty = I_8$ does not determine the SQFT uniquely. Two distinct choices for $\mathcal{Z}^{[1]}$, either $\mathbb{Z}_2$ or trivial. This gives E_1 or $\tilde{E}_1$.
- The case E_1 is special, with $\Phi = \mathbb{Z}_4$ and $\mathcal{Z}^{[1]} = \mathbb{Z}_2$, with:

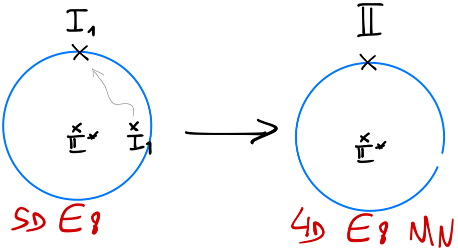
  $$\mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathscr{F} = \mathbb{Z}_2$$

  so that $G_F = SO(3)$.
- All other theories have $\mathcal{Z}^{[1]} = 0$, and thus $\Phi_{\mathrm{tor}} = \mathscr{F}$.

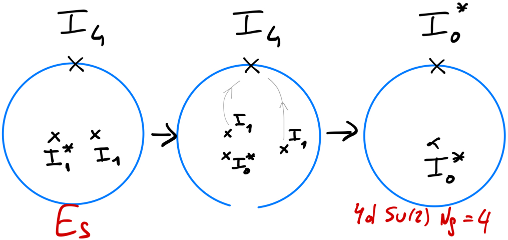
RG flows to 4d

Two types of flows:

- **"zooming in":**
  Here we just decouple the KK scale.



- **"geometric engineering limit":**
  We decouple the KK scale *and* the instanton particles.



Modularity of the U-plane

In many interesting special limits, **the U-plane is a modular curve:**

$$\overline{\mathcal{B}} \cong \mathbb{H}/\Gamma \ , \qquad\qquad \Gamma \subset SL(2,\mathbb{Z})$$

This means, in particular, that the mirror map is a modular function:

$$a = a(U) \qquad \leftrightarrow \qquad U = U(\tau)$$

Example: the massless curves:

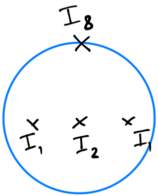
$$\begin{array}{lcll}
E_7 & : & III^* \oplus I_1 & : \ \Gamma^0(2) \\
E_6 & : & IV^* \oplus I_1 & : \ \Gamma^0(3) \\
E_5 & : & I_1^* \oplus I_1 & : \ \Gamma^0(4) \\
E_4 & : & I_5 \oplus I_1 \oplus I_1 & : \ \Gamma^1(5) \\
E_3 & : & I_3 \oplus I_2 \oplus I_1 & : \ \Gamma^0(6) \\
E_1 & : & I_2 \oplus I_1 \oplus I_1 & : \ \Gamma^0(8) \\
E_0 & : & I_1 \oplus I_1 \oplus I_1 & : \ \Gamma^0(9)
\end{array}$$

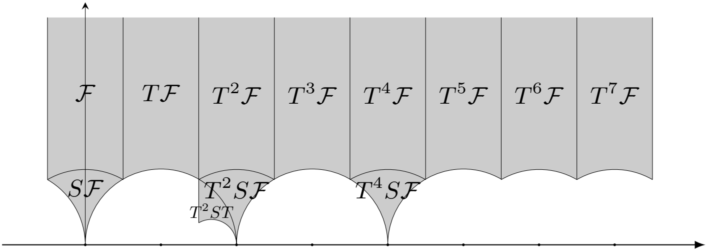
The massless E_8, E_2 and $\tilde{E}_1$ are not modular.

Example: the massless E_1 theory

This is "5d pure $SU(2)_0$ at infinite coupling."

The CB of the massless is a modular curve for the congruence subgroup $\Gamma^0(8)$:





Singularities and monodromies:

$$M_{(-2)} = STS^{-1} \, , \qquad M_{(0)} = (T^2S)T^2(T^2S)^{-1} \, , \quad M_{(-2)} = (T^4S)T(T^4S)^{-1} \, .$$

- At $U = -2$, the monopole $(1, 0)$ is massless, $a_D \to 0$. "Conifold point."
- At $U = 0$, two dyons $(-1, 2)$ are massless.
- At $U = 2$, the dyon $(1, -4)$ is massless.

5d BPS quivers

BPS quivers of 4d KK theories

Given any 4d $\mathcal{N}=2$ field theory $\mathcal{T}$, are hard question is to compute the **BPS spectrum** $\mathscr{S}_u$ at $u \in \mathcal{B}$.

In principle, one can proceed in two steps:

- Identify the BPS category $\mathscr{T}_{\mathcal{T}}^{\mathrm{BPS}}$ of $\mathcal{T}$.
- Identify the **stable objects** in $\mathscr{T}_{\mathcal{T}}^{\mathrm{BPS}}$.

In physics language, it's a F-term/D-term dichotomy.

For our 5d theories on a circle,

$$\mathcal{T} = D_{S^1}\mathcal{T}_{\mathbf{X}}^{\mathrm{5d}}$$

the BPS states are D0/D2/D4 bound states in IIA.

The BPS category of the KK theory is the derived category of coherent sheaves on the resolved singularity $\tilde{\mathbf{X}}$:

$$\mathscr{T}^{\mathrm{BPS}}_{D_{S^1}\mathcal{T}_{\mathbf{X}}^{\mathrm{5d}}} = \mathrm{D}^b(\mathrm{coh}\tilde{\mathbf{X}})$$

Π-stables branes are the stables objects that give us the BPS spectrum. [Douglas, Fiol, Romelsberger, 2000]. They are (essentially) counted by the DT invariants of $\tilde{\mathbf{X}}$. (See [Duan, Ghim, Yi, 2020] for an important caveat.)

BPS quivers of 4d KK theories

There often exists a quiver description of the BPS states. [Alim, Cecotti, Cordova, Espahbodi, Rastogi, Vafa, 2011]

Let $\mathcal{A}_Q$ be the Jacobian algebra of (Q, W). We then have:

- $\mathscr{T}_T^{\mathrm{BPS}} = \mathrm{D}(\mathcal{A}_Q\text{-mod})$.
- The BPS states are given by (quantising the moduli spaces of) the θ-stable representations.

For the KK theories of interest, we call this the **the 5d BPS quiver**. In the physics literature, it is best known as the fractional-brane quiver of the canonical singularity $\mathbf{X}$ (if $\mathbf{X}$ admits a crepant resolution), and as non-commutative crepand resolution (NCCR) of $\mathbf{X}$ in the maths literature. One expects:

$$\mathrm{D}^b(\mathrm{coh}\tilde{\mathbf{X}}) \cong \mathrm{D}(\mathcal{A}_Q\text{-mod})$$

Various techniques exists to extract the quiver (and superpotential) (Q, W) from the B-model on $\tilde{\mathbf{X}}$ – see *e.g.* [CC, Del Zotto, 2019].

Here, we would like to directly derive Q from the type IIB mirror – *i.e.* from the SW geometry.

5d BPS quivers from the U-plane: simple prescription

We focus again on rank-one theories.

Basic idea: consider a CB configuration with only I_k singularities at $u_{*,i}$.
Then, motivated by the IIB and F-theory picture, we:

- $\mathrm{conjecture}$ that the BPS spectrum consists of charges (m, q) that are generated by the dyons $\gamma_i = (m_i, q_i)$ that become massless at $u_{*,i}$;
- to each I_k, we associate k quiver nodes;
- the number of arrows between nodes, $(i) \to (j)$, is given by the Dirac pairing:

$$n_{ij} = \langle \gamma_i, \gamma_j \rangle = \det \begin{pmatrix} m_i & q_i \\ m_j & q_j \end{pmatrix}$$

Comments:

- This pedestrian method does not gives us W.
- In simple cases, we can prove that this quiver description exists by computing the central charges Z_γ near the origin of the CB.

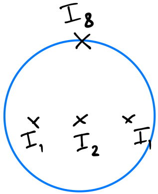
Example: the massless E_1 theory

Consider the E_1 theory. Recall that we have the light dyons:

$$(1,0)\ , \qquad (-1,2)\times 2\ , \qquad (1,-4)\ .$$

This gives the quiver:

$$\begin{tikzcd}
\mathcal{E}_{\gamma_1=(1,0)} \arrow[r, two heads] \arrow[d, two heads] & \mathcal{E}_{\gamma_2=(-1,2)} \arrow[d, two heads] \\
\mathcal{E}_{\gamma_3=(-1,2)} \arrow[r, two heads] & \mathcal{E}_{\gamma_4=(1,-4)} \arrow[ul]
\end{tikzcd}$$



Indeed, the E_1 geometry is the well-known local $\mathbb{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$, and the quiver above is a known 'toric' quiver for this geometry. It is valid at $U = 0$ on the massless CB.

Modular curves and quiver points

In practise, we use modularity to identify the light dyons. We can classify all modular CB configurations for any of the rank-one theories.

For instance, for $D_{S^1}E_8$ and restricting to congruence subgroups (for simplicity):

$\{F_v\}$	$\mathrm{rk}(\Phi)$	Φ_{tor}	$\mathfrak{g}_F$	$\Gamma \in \mathrm{PSL}(2,\mathbb{Z})$
I_1, I_2, III^*	0	$\mathbb{Z}_2$	$E_7 \oplus A_1$	$\Gamma_0(2)$
I_1, I_3, IV^*	0	$\mathbb{Z}_3$	$E_6 \oplus A_2$	$\Gamma_0(3)$
$2I_1, I_4^*$	0	$\mathbb{Z}_2$	D_8	$\Gamma_0(4)$
I_1, I_4, I_1^*	0	$\mathbb{Z}_4$	$D_5 \oplus A_3$	$\Gamma_0(4)$
$2I_1, 2I_5$	0	$\mathbb{Z}_5$	$A_4 \oplus A_4$	$\Gamma_1(5)$
I_1, I_6, I_3, I_2	0	$\mathbb{Z}_6$	$A_5 \oplus A_2 \oplus A_1$	$\Gamma_0(6)$
$2I_1, I_8, I_2$	0	$\mathbb{Z}_4$	$A_7 \oplus A_1$	$\Gamma_0(8)$
$3I_1, I_9$	0	$\mathbb{Z}_3$	A_8	$\Gamma_0(9)$
I_1, III^*, II	1	$-$	E_7	$PLS(2,\mathbb{Z})$
I_1, III, IV^*	1	$-$	$E_6 \oplus A_1$	$PLS(2,\mathbb{Z})$
I_1, I_2^*, III	1	$\mathbb{Z}_2$	$D_6 \oplus A_1$	$\Gamma_0(2)$
I_1, I_3^*, II	1	$-$	D_7	$\Gamma_0(3)$
$I_1, I_5, 2III$	2	$-$	$A_4 \oplus 2A_1$	$\Gamma_0(5)$
$I_1, I_7, 2II$	2	$-$	A_6	$\Gamma_0(7)$

Modular curves and quiver points

One can then identify the light particles and, in favourable cases, the 5d BPS quiver.

Example: The $D_{S^1}E_8$ CB configuration $\mathcal{S} = (I_1, I_6, I_3, I_2)$, with:

$$\mathscr{S} \; : \qquad I_6 : \; 6(1,0) \; , \qquad I_2 : \; 2(-3,1) \; , \qquad I_3 : \; 3(2,-1) \; ,$$

$$\begin{tikzcd}
\mathcal{E}_{\gamma_{1,2,3,4,5,6}=(1,0)} \arrow[r] & \mathcal{E}_{\gamma_{7,8}=(-3,1)} \arrow[dl] \\
\mathcal{E}_{\gamma_{9,10,11}=(2,-1)} \arrow[u] &
\end{tikzcd}$$

This is a correct 3-blocks quiver for dP_8, which can be obtained from B-branes [Wijnholt, 2002; Karpov, Nogin, 1997]. Here, we derived it from the mirror.

Note: By removing γ_1, we get a BPS quiver for the 4d E_8 MN theory.

Summary and outlook

Summary:

- We revisited a general approach to rank-one 4d $\mathcal{N}=2$ SQFT in terms of rational elliptic surfaces.
- We pointed out that the Persson classification of RES gives classification of CB configurations.
- We determined the flavour symmetry group directly from the SW geometry.
- We discussed **the Coulomb branch physics of 5d SCFTs** on $\mathbb{R}^4 \times S^1$.
- We studied global properties of the U-plane, such as modularity.

Outlook:

- We initiated a study of quiver points on the U-plane. More systematic analysis needed.
- These elementary considerations are fundamental to a better understanding of partition functions of 5d SCFTs on five-manifolds. *Work in progress.*