Quivers, Affine Symmetries, Wall-Crossing

2101.01681
2107.14255 with F. Del Monte

Goals:

- BPS states of M-theory on local CY3
- instanton particles + monopole strings in 5d $\mathcal{N}=1$ QFT
- rank-0 DT invariants

Results:

Exact, explicit answers for local Surfaces

$$S = \mathbb{F}_0 \,,\ dP_n \qquad n = 3, 5$$

$\mathcal{X}$ isolated CY3 sing.ty

M-theory on $\mathcal{X} \times \underline{\underline{\mathbb{R}^{1,4}}}$ $\rightsquigarrow$ 5d $\mathcal{N}=1$ SCFT

$\pi : X \to \mathcal{X}$ $\rightsquigarrow$ deformation to QFT

Kähler moduli space $\rightsquigarrow$ $\begin{cases} \text{Coulomb branch} \\ \text{masses for flavor} \end{cases}$

$$\begin{cases} \dim H_2(X,\mathbb{Z}) = \underline{r+f} \\ \dim H_4(X,\mathbb{Z}) = r \end{cases}$$

$r =$ rank of 5d Coulomb br.

$f =$ rank of G_F

M-thy on $X \times S^1 \times \mathbb{R}^4 \cong$ Type IIA on $X \times \mathbb{R}^4$

$\rightsquigarrow$ $\boxed{\underline{4d\ \mathcal{N}=2}\ \text{KK}\quad \text{QFT}\quad \text{on}\ \mathbb{R}^4}$

cplx Kähler moduli $\longrightarrow$ cplx Coulomb & masses

BPS states

$$\left\{\begin{array}{l} M2 \text{ on } \mathcal{C}_2 \times \mathbb{R} \\ M5 \text{ on } \mathcal{C}_4 \times S^1 \times \mathbb{R} \end{array}\right. \rightsquigarrow$$

$\underline{\text{IIA}}$

D0 on $\underline{\underline{\{pt\}}} \times \underline{\underline{\mathbb{R}}}$

D2 on $\underline{\underline{\mathcal{C}_2}} \times \underline{\underline{\mathbb{R}}}$

D4 on $\underline{\underline{\mathcal{C}_4}} \times \underline{\underline{\mathbb{R}}}$

particles

charge lattice $\Gamma = (\underset{=}{\overset{1}{H_0}} \oplus \underset{=}{\overset{r+f}{H_2}} \oplus \underset{=}{\overset{r}{H_4}})(X, \mathbb{Z})$

$\cong \mathbb{Z}^{\underbrace{2r}_{\text{e.m.}} + \underbrace{f}_{\text{flavor}} + \underbrace{1}_{\text{KK or D0 charge}}}$

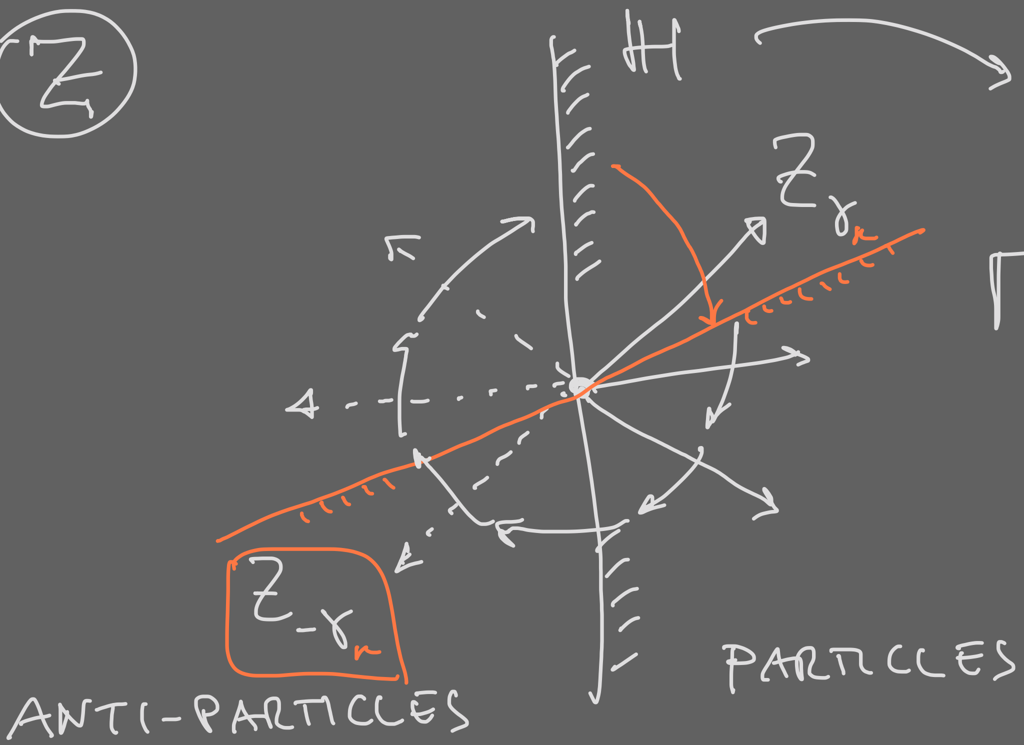
<u>Stability condition</u>

$\gamma \longmapsto \boxed{\mathcal{H}_\gamma^{BPS}(Z)} \qquad \gamma \in \Gamma$

central charge $\underset{=}{Z} \in \mathrm{Hom}(\Gamma, \mathbb{C})$

<u>EXTENDED $\mathcal{M}_{stab.}$</u> $\cong \mathbb{C}^{2r+f+1}$

Fix Z



CHOICE

$$\Gamma = \underbrace{\Gamma_+} \cup \underbrace{\Gamma_-} \cup \{0\}$$

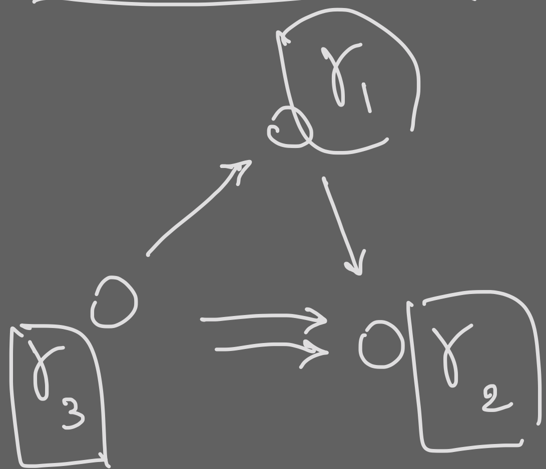
P.S.C. or motivic D.T.

spin(3) $\quad$ $SU(2)_R$

$$\boxed{\Omega(\gamma, y, Z)} := \mathrm{Tr}_{\mathcal{H}^{BPS}_{\gamma}(Z)}\, y^{2J_3} (-y)^{2I_3}$$

$$= \sum_s \boxed{a_s(\gamma)} \cdot (-y)^s$$

BPS Quivers



$2r+f+1$

nodes $\longleftrightarrow$ generators γ_i of $\boxed{\Gamma_+}$ $\boxed{\mathbb{H}}$

arrows $\longleftrightarrow$ $\langle \gamma_i, \gamma_j \rangle = 2$ Dirac Schwinger Zwanziger

$\boxed{\gamma} = \sum N_i \gamma_i$, $N_i \geq 0$ $\rightsquigarrow$ 1d $\mathcal{N}=4$ SQM $\begin{cases} G = \prod_i U(N_i) \\ \text{matter: chiral multiplets for each arrow} \end{cases}$

Higgs branch

$$\boxed{\mathcal{M}_\gamma(z)} = \left\{ B^a_{ij} \;\middle|\; \frac{\partial W}{\partial B^a_{ij}} = 0, \; \text{D-term eqs} \right\} / G$$

$$\mathcal{H}^{BPS}_\gamma(Z) = H^*(\mathcal{M}_\gamma(Z))$$

Important exple: $\gamma = \gamma_i$

$$\mathcal{M}_{\gamma_i} = \{pt.\} \Rightarrow \boxed{\Omega(\gamma_i, y) = 1} \quad \text{“}\forall Z\text{”}$$

Rotating $\mathbb{H}$ changes Γ_+ $\rightsquigarrow$ change Q by $\boxed{\text{MUTATION}}$

$$\boxed{\mu_k(\gamma_j)} = \begin{cases} -\gamma_j & \text{if } j = k \\ \gamma_j + [C_{jk}]_+ \gamma_k & \text{if } j \neq k \end{cases}$$

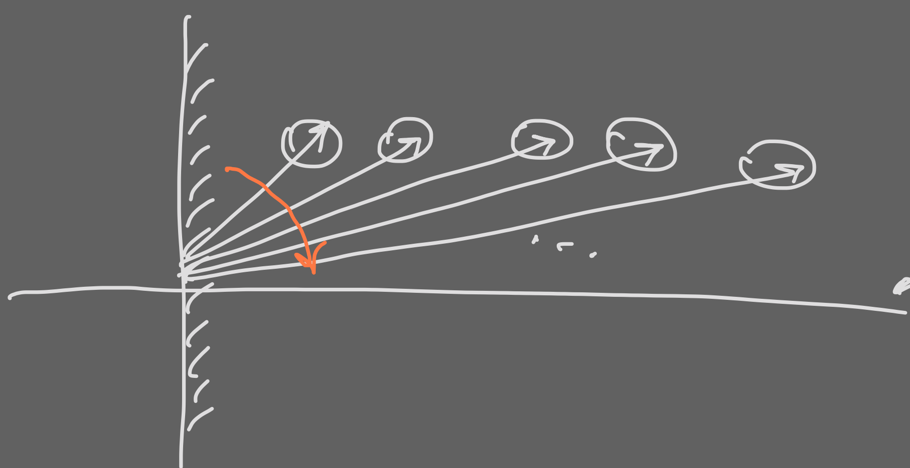
adjacency

Multation Algo. (Alim et. al. '11)

$$Q \xrightarrow{\mu} Q' \xrightarrow{\mu} Q'' \longrightarrow \cdots$$

$$\{\gamma_i\} \longrightarrow \{\gamma_i'\} \longrightarrow \{\gamma_i''\} \longrightarrow \cdots$$

$$\Omega(\gamma_i) = \Omega(\gamma_i') = \Omega(\gamma_i'') = \cdots = 1$$



ACCUMULATION RAY

Quiver Symmetries

$\mathcal{H}_\gamma^{BPS}(Z)$ and $\Omega(\gamma, y)$ is piecewise constant in Z

$$\Phi(x) = \prod_{n \geq 0} (1 + x y^{2n+1})^{-1}$$

$$X_\gamma X_{\gamma'} = y^{\langle \gamma, \gamma' \rangle} X_{\gamma+\gamma'}$$

$$\boxed{\Omega(\gamma, y)} = \sum_s a_s(\gamma) (-y)^s$$

$\boxed{\mathrm{Arg}\, Z}$

$$\mathbb{U} = \prod_\gamma \prod_{s \in \mathbb{Z}} \Phi((-y)^s X_\gamma)^{\boxed{a_s(\gamma)}}$$

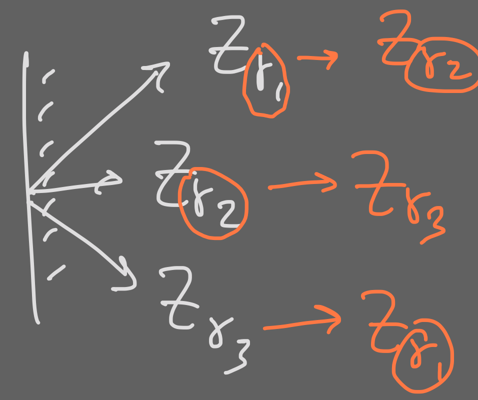
WCF of Kontsevich - Soibelman

$\boxed{\mathbb{U}}$ is invt. under changes of Z

$\Rightarrow$ $\mathbb{U}$ determines full BPS spectrum at any

$Z \in \mathcal{M}_{stab.}$

$$Q: \quad \begin{tikzcd} \gamma_1 \arrow[rr] & & \gamma_2 \arrow[dl] \\ & \gamma_3 \arrow[ul] & \end{tikzcd} \qquad \Pi_Q \simeq \mathbb{Z}_3$$



$\pi \in \Pi_Q$ $\qquad \pi(\gamma_i) = \gamma_{i+1 \bmod 3}$ $\qquad Q$ is unchanged

<u>BUT</u>: $\qquad \pi(Z) \neq Z$

$$\Omega(\gamma, y, Z) \neq \Omega(\gamma, y, \pi(Z))$$

$$\shortparallel$$

$$\Omega(\pi(\gamma), y, \pi(Z))$$

at Z $\qquad\qquad$ at $\pi(Z)$

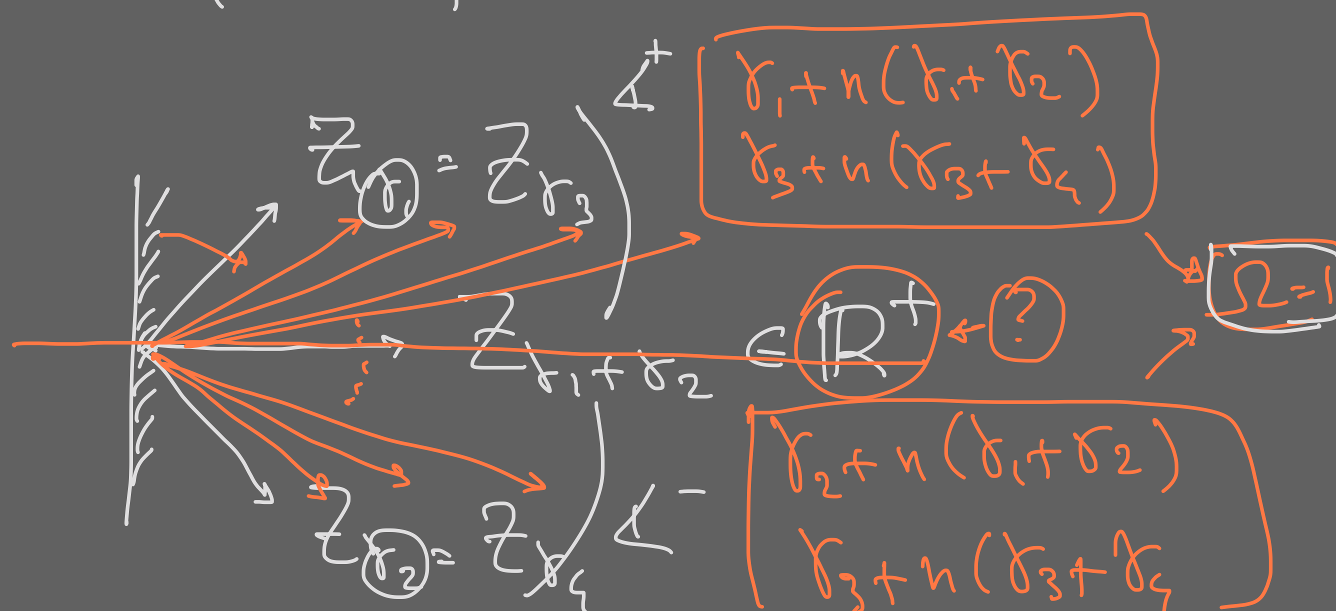
$$\boxed{\mathbb{U}} = \overleftarrow{\prod_{\gamma,s}} \Phi\left((-y)^s X_\gamma\right)^{\boxed{a_s(\gamma)}} \equiv \boxed{\mathbb{U}'} = \overleftarrow{\prod_{\gamma,s}}' \Phi\left((-y)^s X_\gamma\right)^{\boxed{a_s(\gamma)}}$$

Ex: local $\mathbb{F}_0$

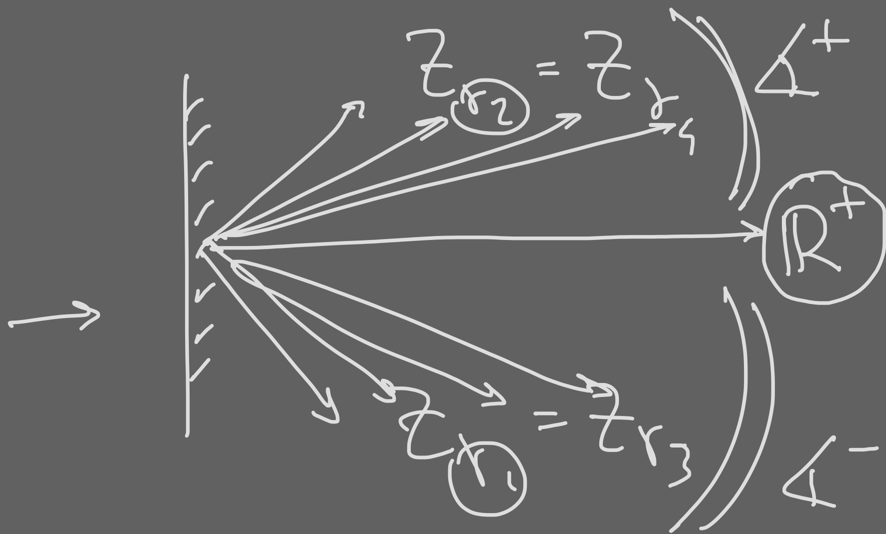
$$\begin{tikzcd} \overset{\gamma_1}{\circ} \arrow[r, shift left] \arrow[r, shift right] & \overset{\gamma_2}{\circ} \arrow[d, shift left] \arrow[d, shift right] \\ \underset{\gamma_4}{\circ} \arrow[u, shift left] \arrow[u, shift right] & \underset{\gamma_3}{\circ} \arrow[l, shift left] \arrow[l, shift right] \end{tikzcd}$$

$$\Pi_Q \cong \mathbb{Z}_4$$

Closset - Del Zotto



$\pi \in \mathbb{Z}_4$ $\qquad \gamma_i \to \gamma_{i+1}$



$\gamma_2 + n(\gamma_2 + \gamma_3)$

$\gamma_4 + n(\gamma_4 + \gamma_1)$

$\gamma_1 + n(\gamma_4 + \gamma_1)$

$\gamma_3 + n(\gamma_3 + \gamma_2)$

$$\mathbb{U} = \underbrace{\mathbb{U}(\Delta^+)}_{\checkmark} \cdot \mathbb{U}(\mathbb{R}^+) \cdot \underbrace{\mathbb{U}(\Delta^-)}_{\checkmark}$$

$$\mathbb{U}(\mathbb{R}^+) = \sum_\gamma c_\gamma X_\gamma$$

$\|$

$$\mathbb{U}' = \underbrace{\mathbb{U}'(\Delta^+)}_{\checkmark} \cdot \underline{\mathbb{U}'(\mathbb{R}^+)} \cdot \underbrace{\mathbb{U}'(\Delta^-)}_{\checkmark}$$

$$\mathbb{U}'(\mathbb{R}^+) \to \boxed{\sum_\gamma c_{\pi^{-1}(\gamma)} X_\gamma}$$

$$\boxed{\mathbb{U}(\mathbb{R}^+)} = \prod_{\substack{k \geq 0 \\ s = \pm 1}} \Big[\Phi\big((-y)^s X_{\underbrace{\gamma_1+\gamma_2+k\gamma_{D0}}_{D2_f - D0}}\big)^{-1} \cdot \Phi\big((-y)^s X_{\underbrace{\gamma_3+\gamma_4+k\gamma_{D0}}_{\overline{D2}_f - D0}}\big)^{-1} \Big]$$

$$\gamma_{D0} = \gamma_1+\gamma_2+\gamma_3+\gamma_4$$

$$\gamma_1+\gamma_2 = D2_f$$

<u>pure D0 branes</u>: (Mozgovoy - Pioline)

$$\boxed{\Omega(k\gamma_{D0}, y) = y^3 + 2y + y^{-1}}$$

- WCF of KS ✓
- Quiver Sym.
- "smart" choice of Z $\rightsquigarrow$ Collimation Chamber

ONLY ACCUMULATION RAY is on $\mathbb{R}^+$

(Iqbal Neitzke Vafa
Hanany - Iqbal)

$$\Gamma = \Gamma_g \oplus \boxed{\Gamma_f}$$

$\Gamma_f \cong$ affine root lattice of $\hat{g}$ $\circlearrowleft$

$$W(\hat{g}) \subset \boxed{\widetilde{W}(\hat{A})}$$

$$\boxed{\widetilde{W}(\hat{A})} \cup \boxed{Aut_Q} = \left\{ \begin{array}{l} \text{mutations} \\ \text{perms.} \\ \text{inversion} \end{array} \right.$$

extends to Γ $\longleftarrow$ $\boxed{Aut_Q}$

Mizuno

Bershtein
Gavrilenko
Marshakov

Affine translations   Bonelli Del Monte Tanzini

- repr. by mutations
- tilting or rot. of $\mathcal{H}$ by Z
- $Z \in$ collimation chambers

$\rightarrow$ $\underline{\underline{\mathbb{F}_0}}$, $\underline{\underline{dP_3}}$, $\underline{\underline{dP_5}}$ $\rightsquigarrow$ EXACT expr. for $\mathcal{U}$

$\Rightarrow$ ALL $\Omega(\gamma, y)$

$\forall Z$