Operator growth in 2d CFT

Shouvik Datta

CERN

based on arXiv:2110.10519 with Pawel Caputa

Introduction and motivation

Thermal relaxation in quantum systems

The process of relaxation to thermal equilibrium is a part of our everyday experience.

However, it is not always clear how macroscopic phenomena emerge microscopic/quantum-mechanical details.

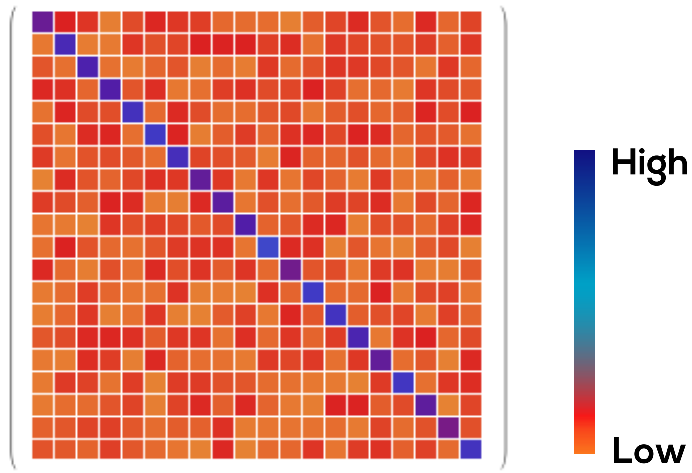
Microscopic laws are time-reversal invariant.
But, thermodynamic laws aren't.

How does this irreversible behaviour emerge microscopically?

Microstates and thermalization

Criterion: Eigenstate Thermalization Hypothesis (ETH)

$$\langle O \rangle_\beta = \langle E_n | O | E_n \rangle$$



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A single, pure, energy eigenstate may be good enough to reproduce thermodynamics.

[Deutsch, Srednicki]

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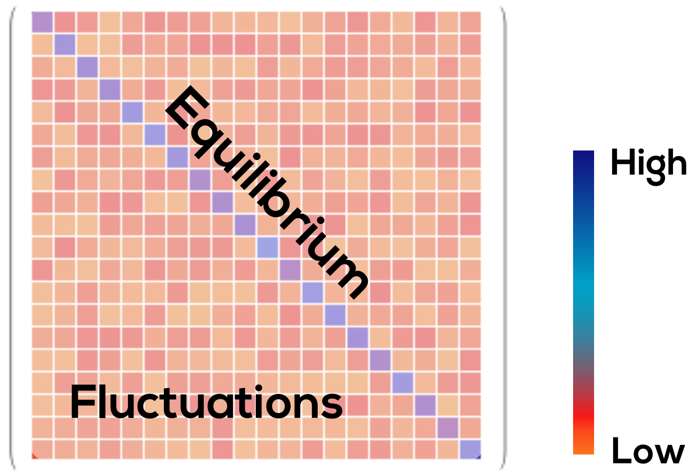

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[Deutsch, Srednicki]

Spreading of quantum information

Another route to understand thermalization:
explore how information/entanglement spreads in a system.

Information once stored in simple local operators scrambles into a large number of degrees of freedom at later times.

This phenomenon is called operator growth.
'Simple operators grow into ones with higher complexity.'

[von Keyserlingk-Rakovszky-Pollmann-Sondhi; Nahum-Vijay-Haah; Khemani-Vishwanath-Huse; Roberts-Stanford-Streicher; Qi-Streicher; …]

Operator growth

Consider the Heisenberg evolution of a local operator

$$e^{iHt}\mathcal{O}(0)e^{-iHt} = \mathcal{O}(0) + it[H,\mathcal{O}(0)] - \frac{t^2}{2}[H,[H,\mathcal{O}(0)]] - \frac{it^3}{6}[H,[H,[H,\mathcal{O}(0)]]] + \cdots \ .$$

Let O be a simple local operator and the Hamiltonian has few-body interactions.

However, the effect of the operator spreads throughout the system at late times.

(Figure taken from D. Parker's talk)

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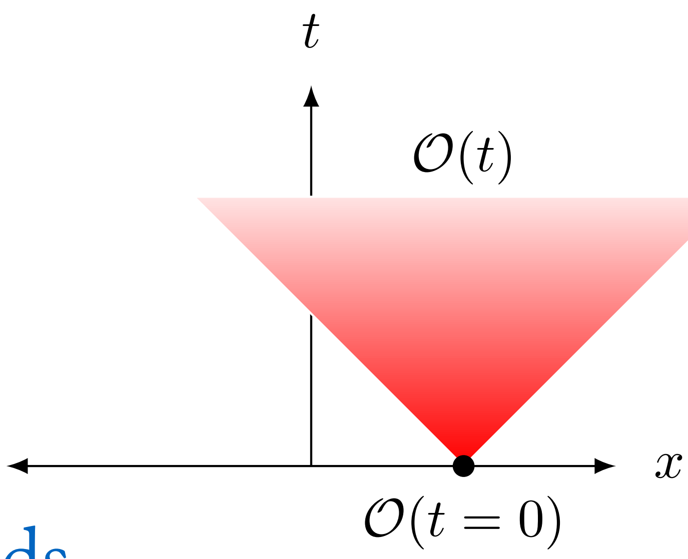

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Operator growth

Consider the Heisenberg evolution of a local operator

$$\mathcal{O}(t) \equiv e^{i\mathcal{L}t}\mathcal{O}(0) \qquad \mathcal{L} = [H, *] \qquad \mathcal{L}^3\mathcal{O} = [H,[H,[H,\mathcal{O}]]]$$

Liouvillian

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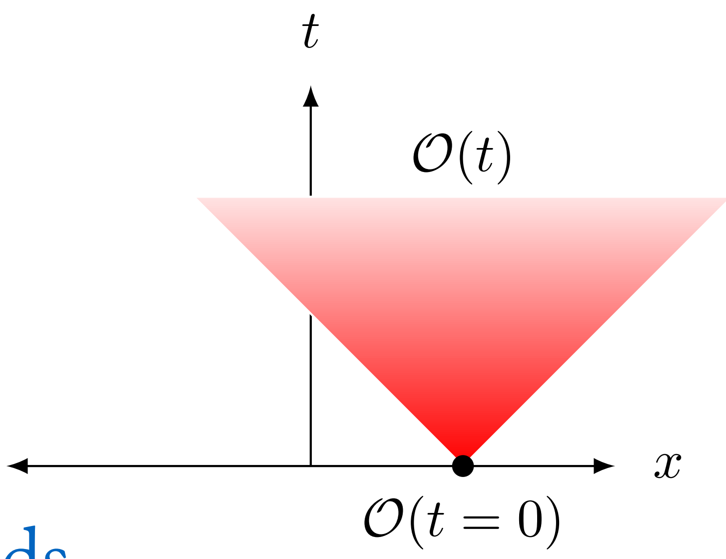

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Operator growth: an example

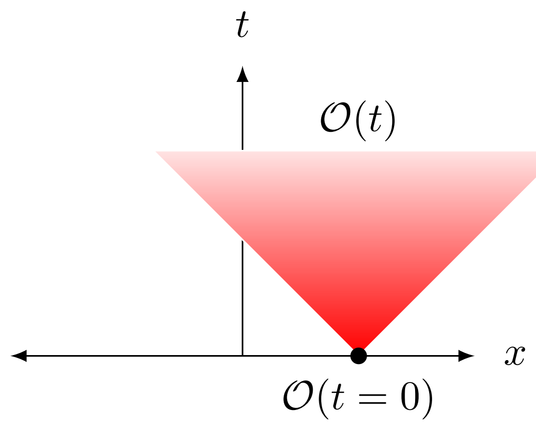
Chaotic Ising model

$$H = \sum_i \left(Z_i \cdot Z_{i+1} + B_x X_i + B_z Z_i \right)$$

$$\mathcal{O} = X_1$$

Liouvillian

$$\mathcal{L} = [H, *]$$



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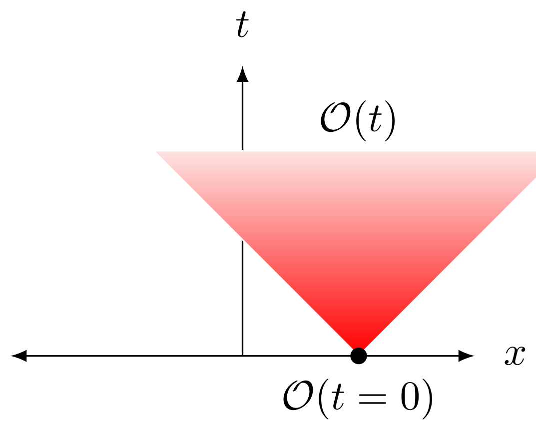

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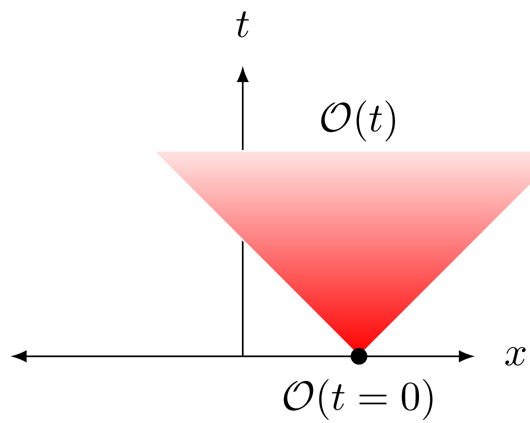
$$\mathcal{O} = X_1 \longleftarrow$$ How does this operator grow?

$$\mathcal{L}O = 2iY_1 B_z + 2iY_1 \cdot Z_2$$ **2 terms**

Liouvillian

$$\mathcal{L} = [H, *]$$



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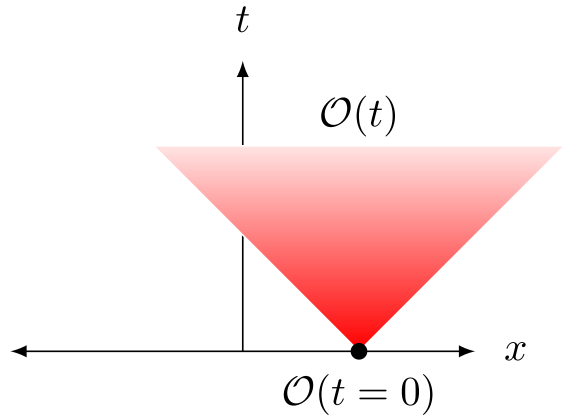
6 terms



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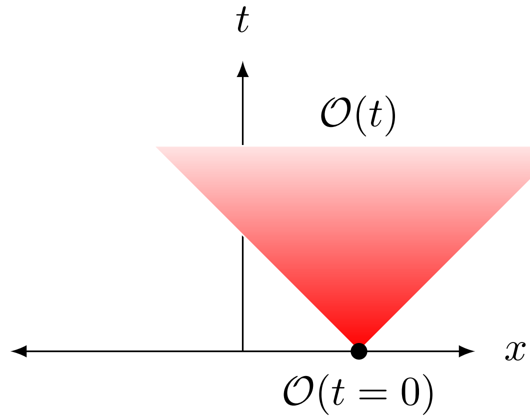

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$$\begin{aligned}\mathcal{L}^2 O = &\, 4B_x Y_1 \cdot Y_2 - 4Z_1 B_x B_z - 4B_x Z_1 \cdot Z_2 \\ &+ 8B_z X_1 \cdot Z_2 + 4X_1 B_z^2 + 4X_1 \cdot Z_2^2\end{aligned}$$

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$$\begin{aligned}\mathcal{L}^3 O = &- 24iB_x B_z X_1 \cdot Y_2 - 8iB_x B_z Y_1 \cdot X_2 - 8iB_x X_1 \cdot Y_2 \cdot Z_2 \\ &- 16iB_x X_1 \cdot Z_2 \cdot Y_2 - 8iB_x Y_1 \cdot X_2 \cdot Z_3 - 8iB_x Y_1 \cdot Z_1 \cdot X_2 \\ &+ 8iY_1 B_x^2 B_z + 16iB_x^2 Y_1 \cdot Z_2 + 16iB_x^2 Z_1 \cdot Y_2 + 24iB_z^2 Y_1 \cdot Z_2 \\ &+ 24iB_z Y_1 \cdot Z_2^2 + 8iY_1 B_z^3 + 8iY_1 \cdot Z_2^3\end{aligned}$$

13 terms

… increasingly large superpositions of Pauli strings.

How to measure this growth?

A concrete way to diagnose this growth is to consider out-of-time-ordered correlators (OTOCs).

$$\langle [\mathcal{O}(t), W(0)]^2 \rangle_\beta \ \sim \ \epsilon e^{\lambda_L t} \qquad\qquad \lambda_L \leq \frac{2\pi}{\beta}$$

This object shows an exponential growth, showing the inability of the evolved operator to commute with other simple operators.

[Maldacena-Shenker-Stanford; Kitaev]

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[Maldacena-Shenker-Stanford; Kitaev]

But.. can we quantify the growth without using additional probes?

The Lanczos algorithm

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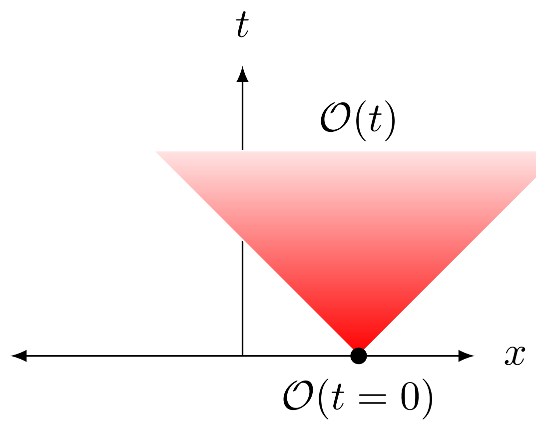

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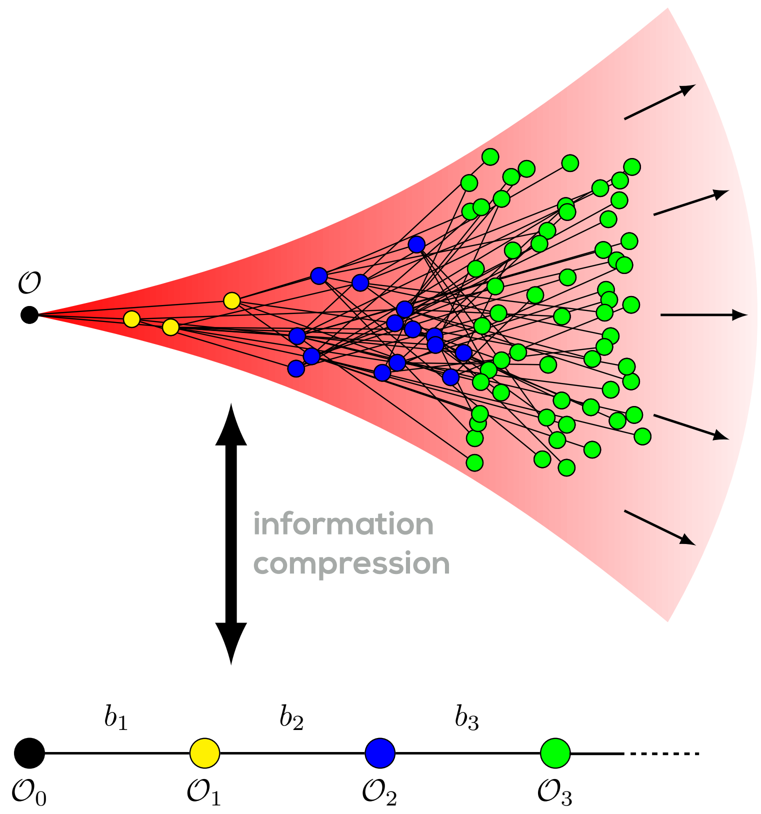
Lanczos algorithm

Note that the Liouvillian evolution gives us a set of operators

$$\{\mathcal{O}, \mathcal{L}\mathcal{O}, \mathcal{L}^2\mathcal{O}, \dots\}$$



(Figure taken from D. Parker's talk)

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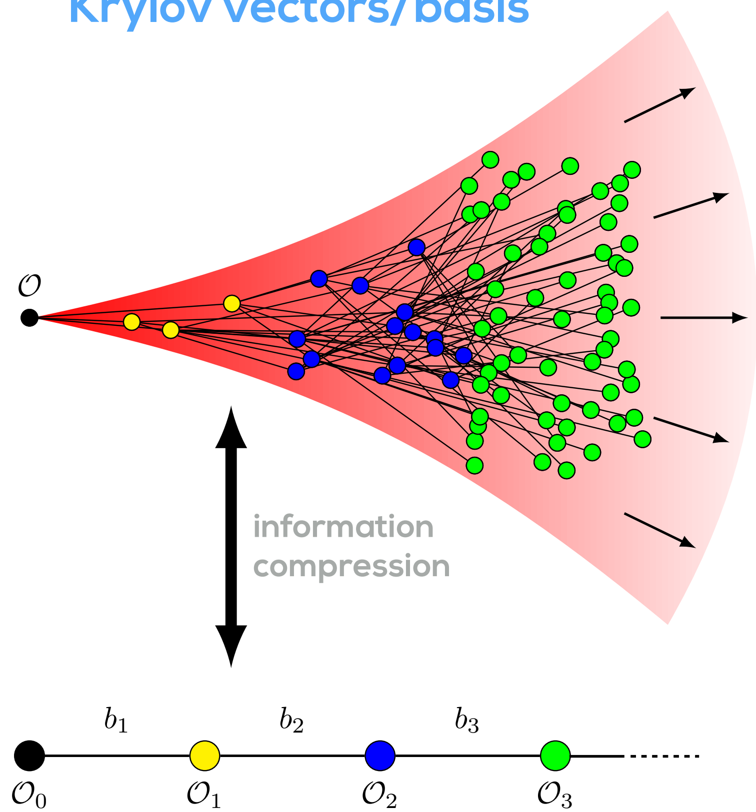
Krylov vectors/basis



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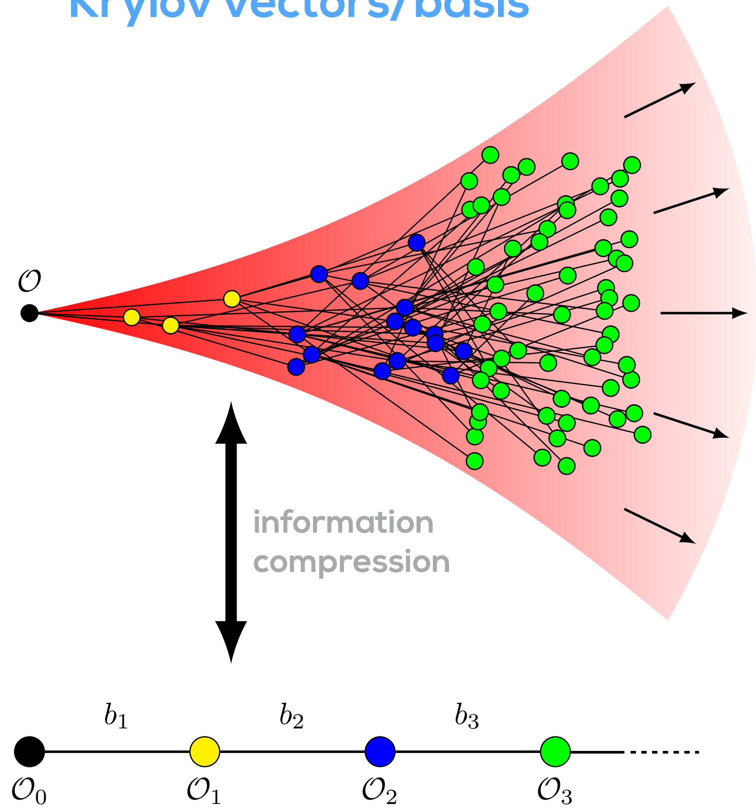

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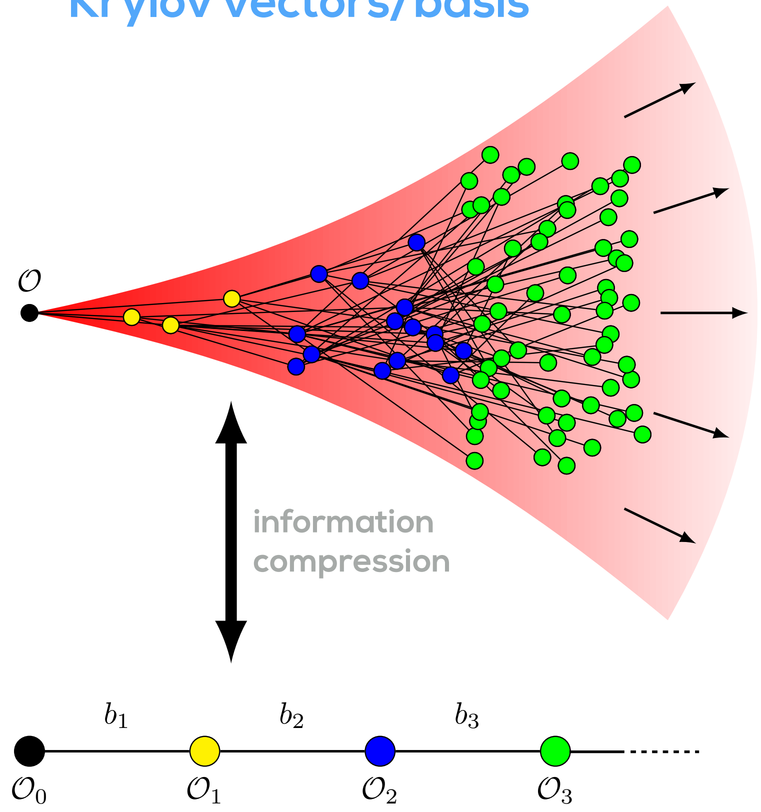
action of the Liouvillian on the Krylov vectors

$$\mathcal{L}|\mathcal{O}_{n-1}) = b_n|\mathcal{O}_n) + b_{n-1}|\mathcal{O}_{n-2})$$

Lanczos coefficients



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Krylov vectors/basis

Liouvillian in the Krylov basis

$$(\mathcal{O}_m^\dagger|\mathcal{L}|\mathcal{O}_n) = \begin{pmatrix} 0 & b_1 & 0 & 0 & \cdots \\ b_1 & 0 & b_2 & 0 & \cdots \\ 0 & b_2 & 0 & b_3 & \cdots \\ 0 & 0 & b_3 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

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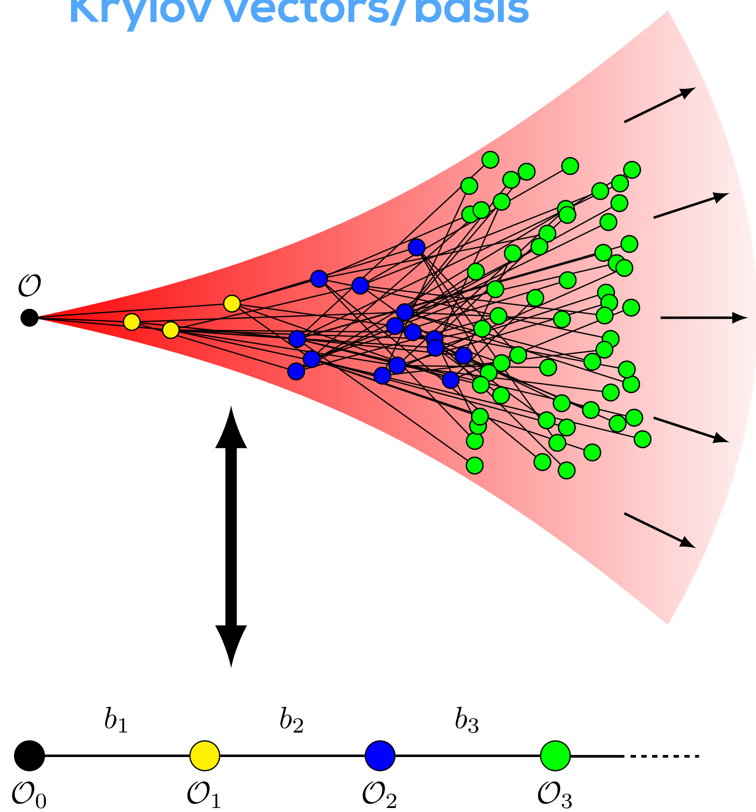

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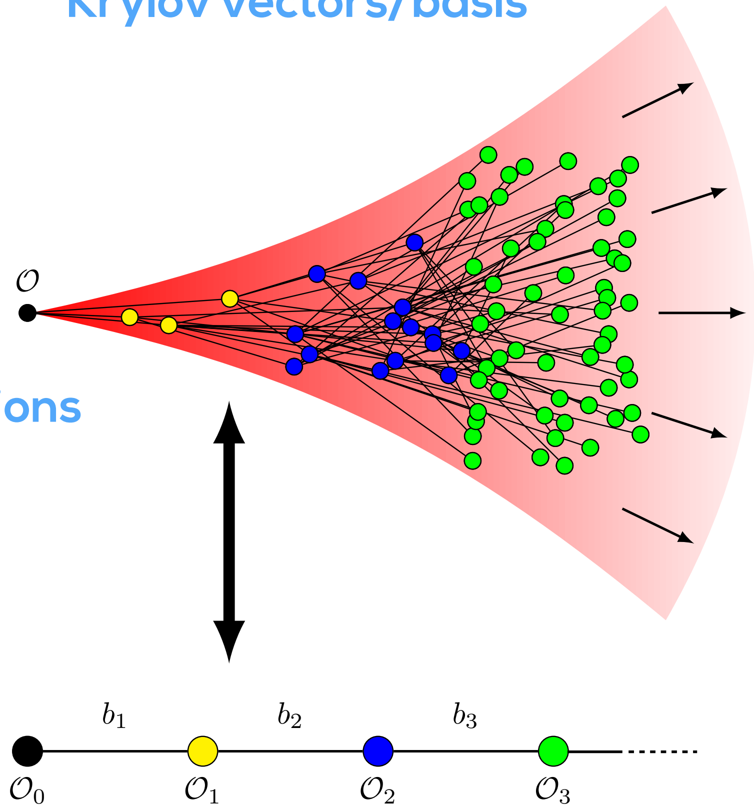
Krylov vectors/basis

The evolved state

$$|\mathcal{O}(t)) = e^{i\mathcal{L}t}|\mathcal{O}) = \sum_n i^n \varphi_n(t)|\mathcal{O}_n)$$

wavefunctions



Discrete Schrödinger equation

$$\partial_t \varphi_n(t) = b_n \varphi_{n-1}(t) - b_{n+1}\varphi_{n+1}(t)$$

$\mathcal{L}|\mathcal{O}_{n-1}) = b_n|\mathcal{O}_n) + b_{n-1}|\mathcal{O}_{n-2})$ $\qquad$ $\varphi_n(t=0) = \delta_{n0}$

Lanczos algorithm: summary

We pick an operator and consider its Liouvillian evolution

$$\{\mathcal{O}, \mathcal{L}\mathcal{O}, \mathcal{L}^2\mathcal{O}, \dots\}$$

$$\{\mathcal{O}_0, \mathcal{O}_1, \mathcal{O}_2, \dots\}$$

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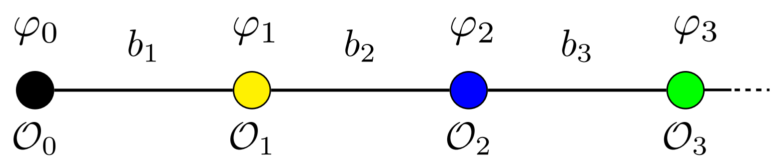

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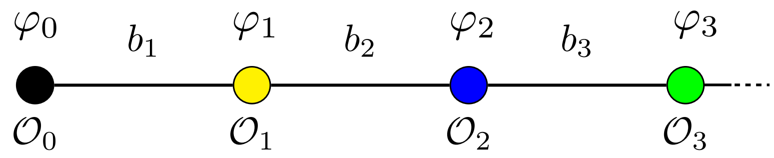
wavefunctions

$$(\mathcal{O}_n^\dagger|\mathcal{L}|\mathcal{O}_{n-1}) = b_n$$

transition amplitudes



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Schrödinger equation

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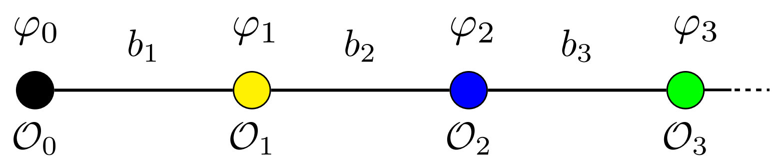

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$\{\mathcal{O}_0, \mathcal{O}_1, \mathcal{O}_2, \dots\}$ **Krylov basis**	$\{b_1, b_2, b_3, \dots\}$ **Lanczos coefficients**	$\{\varphi_0, \varphi_1, \varphi_2, \dots\}$ **wavefunctions**

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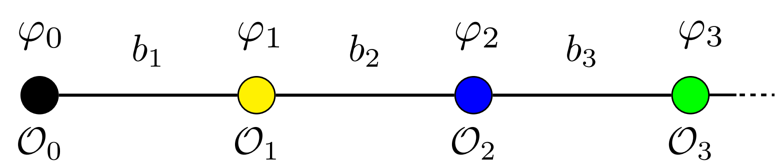

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evolved
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So what?

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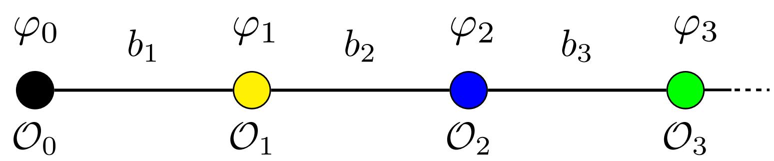

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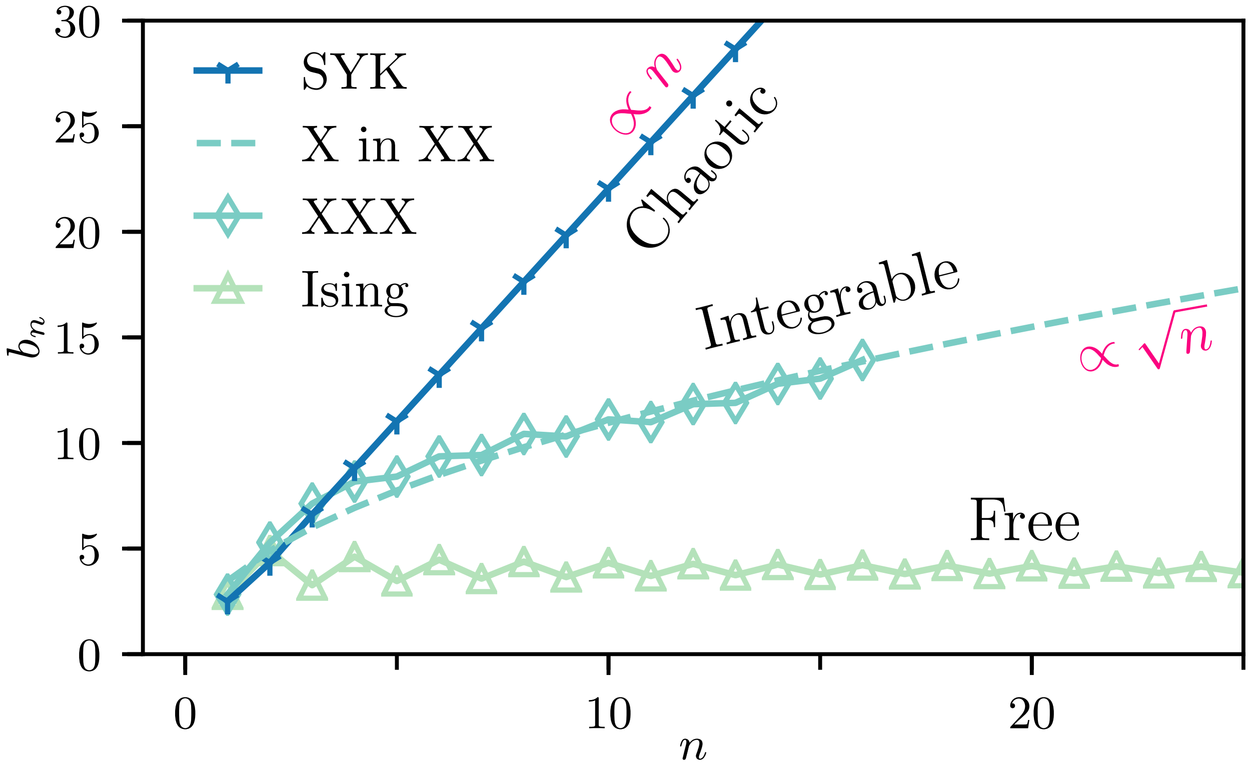
Bounds on operator growth

Growth properties of Lanczos coefficients

Operator growth can be characterized from growth of Lanczos coefficients.



[Parker-Cao-Avdoshkin-Scaffidi-Altman]

Operator growth hypothesis

[Parker-Cao-Avdoshkin-Scaffidi-Altman]

"In a chaotic quantum system, the Lanczos coefficients should grow as fast as possible."

$$\boxed{b_n \leq \alpha\, n + \mathrm{O}(1)}$$

The fastest growth is linear.

This implies an exponential growth of the "size" of the operator.

Analytic evidence

[Parker-Cao-Avdoshkin-Scaffidi-Altman]

Large-q SYK model

$$H_{\text{SYK}}^{(q)} = i^{q/2} \sum_{1 \le i_1 < i_2 < \cdots < i_q \le N} J_{i_1 \ldots i_q} \gamma_{i_1} \gamma_{i_2} \cdots \gamma_{i_q}$$

Single Majorana fermion operator: $\mathcal{O} = \sqrt{2}\gamma_1$

Lanczos coefficients

$$b_n^{\text{SYK}} = \begin{cases} \mathcal{J}\sqrt{2/q} + \mathrm{O}(1/q) & n = 1 \\ \mathcal{J}\sqrt{n(n-1)} + \mathrm{O}(1/q) & n > 1 \end{cases}$$

$\mathcal{J} = \sqrt{q}\, 2^{(1-q)/2} J$

Numerical evidence

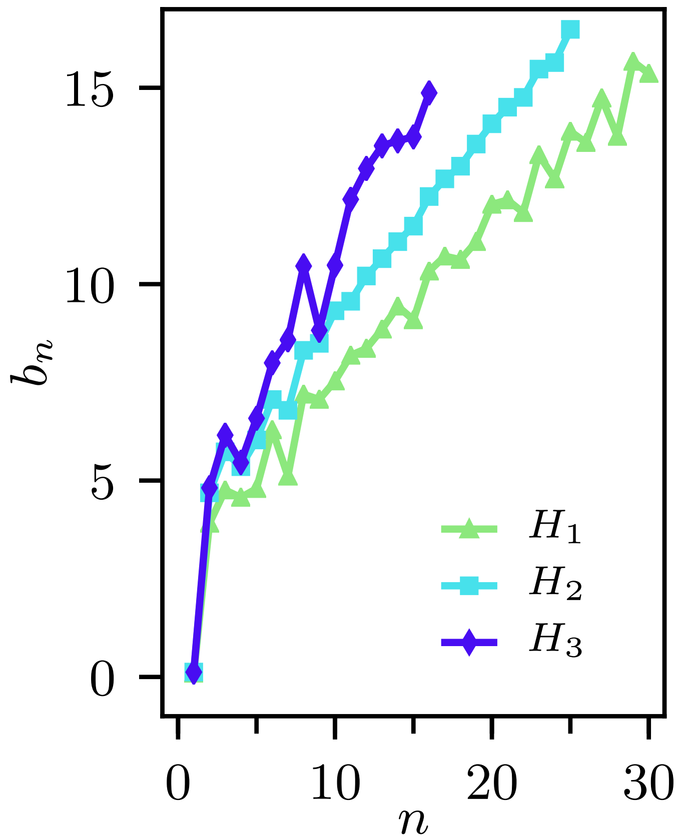
[Parker-Cao-Avdoshkin-Scaffidi-Altman]

Interacting spin-1/2 lattices

$$H_1 = \sum_i X_i X_{i+1} + 0.709 Z_i + 0.9045 X_i$$

$$H_2 = H_1 + \sum_i 0.2 Y_i$$

$$H_3 = H_1 + \sum_i 0.2 Z_i Z_{i+1}$$



Some physical intuition

Schrödinger equation

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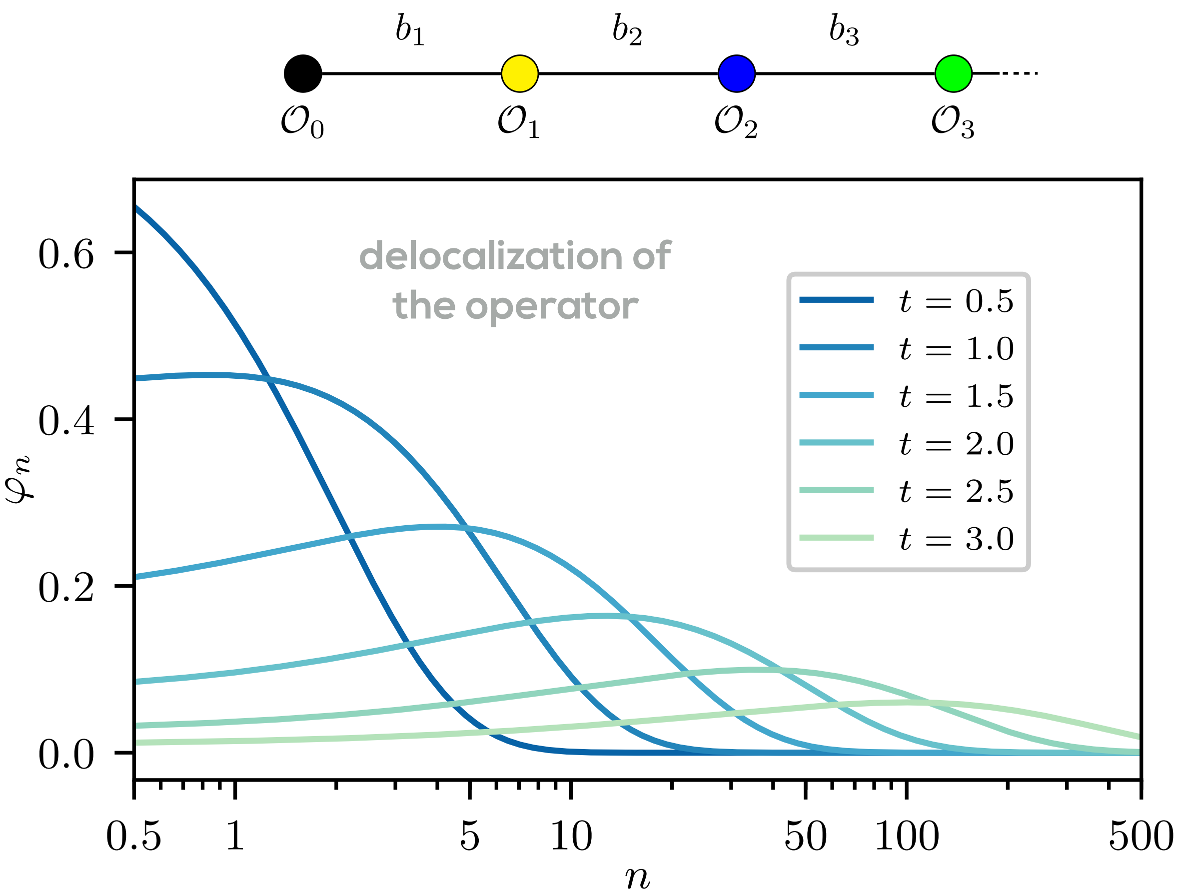

[Parker-Cao-Avdoshkin-Scaffidi-Altman]

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[Barbon-Rabinovici-Shir-Sinha]

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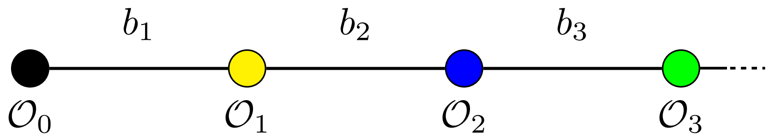


Continuum limit:

$$x = \varepsilon n$$

$$\varphi(x,t) = \varphi_n(t)$$

$$v(x) = 2\varepsilon b(\varepsilon n) = 2\varepsilon b_n$$

This leads to a first-order wave equation

$$\partial_t \varphi(x,t) + v(x)\partial_x \varphi(x,t) + \frac{1}{2} v'(x)\varphi(x,t) = 0$$

Lanczos coefficients provide local velocities for operator spreading.

A measure of operator size

[Parker-Cao-Avdoshkin-Scaffidi-Altman]

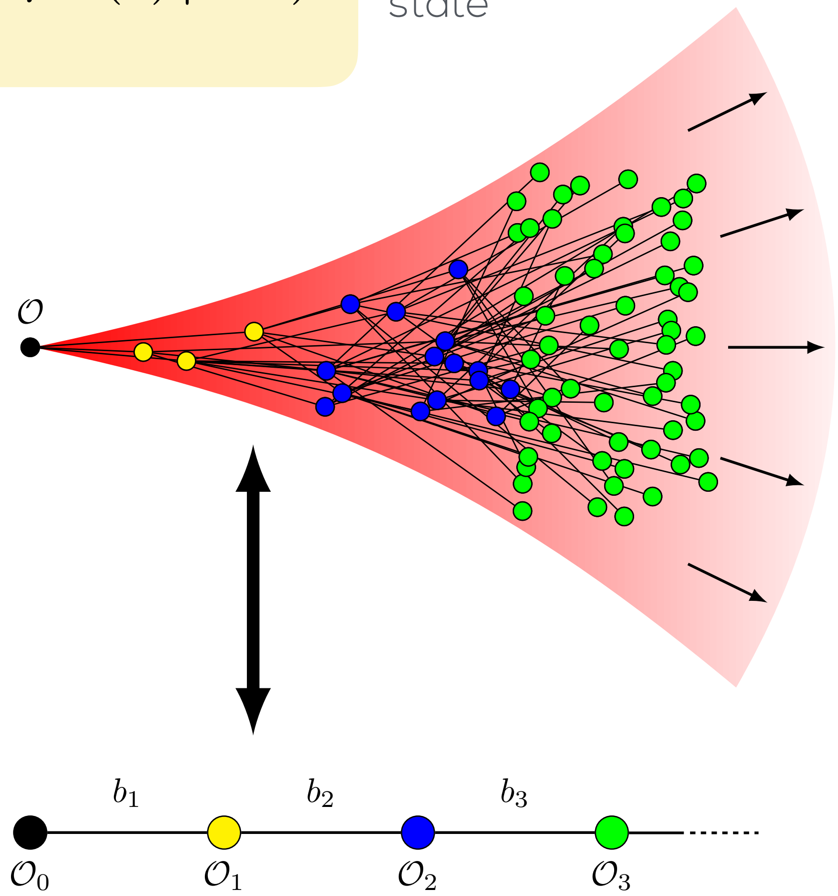
$$|\mathcal{O}(t)) = e^{i\mathcal{L}t}|\mathcal{O}) = \sum_n i^n \varphi_n(t)|\mathcal{O}_n)$$

evolved state

Krylov complexity:
average position in the 1d chain

$$K_{\mathcal{O}}(t) \equiv \sum_n n|\varphi_n(t)|^2$$

This quantity grows exponentially when $b_n \sim \alpha\, n$.



Lanczos algorithm and symmetry

[Caputa-Magan-Patramanis]

action of the Liouvillian on the Krylov vectors

$$\mathcal{L}|\mathcal{O}_{n-1}) = b_n|\mathcal{O}_n) + b_{n-1}|\mathcal{O}_{n-2})$$

Lanczos coefficients

$$(\mathcal{O}_m^\dagger|\mathcal{L}|\mathcal{O}_n) = \begin{pmatrix} 0 & b_1 & 0 & 0 & \cdots \\ b_1 & 0 & b_2 & 0 & \cdots \\ 0 & b_2 & 0 & b_3 & \cdots \\ 0 & 0 & b_3 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

It might be natural to think of the Liouvillian
in terms of ladder operators.

$$\mathcal{L} = \alpha\,(L_+ + L_-)$$

$$\alpha L_+|\mathcal{O}_n) = b_{n+1}|\mathcal{O}_{n+1}), \quad \alpha L_-|\mathcal{O}_n) = b_n|\mathcal{O}_{n-1})$$

SL(2,R) symmetry

[Caputa-Magan-Patramanis]

algebra

$$[L_0, L_{\pm 1}] = \mp L_{\pm 1}$$
$$[L_1, L_{-1}] = 2L_0$$

orthonormal states

$$|h,n\rangle = \sqrt{\frac{\Gamma(2h)}{n!\Gamma(2h+n)}} L_{-1}^n |h\rangle$$

Liouvillian

$$\mathcal{L} = \alpha \left(L_{-1} + L_1\right)$$

action of generators

$$L_0 |h,n\rangle = (h+n) |h,n\rangle \, ,$$
$$L_{-1} |h,n\rangle = \sqrt{(n+1)(2h+n)} \, |h,n+1\rangle \, ,$$
$$L_1 |h,n\rangle = \sqrt{n(2h+n-1)} \, |h,n-1\rangle \, ,$$

SL(2,R) symmetry

[Caputa-Magan-Patramanis]

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$|\mathcal{O}_n)$

Liouvillian

$$\mathcal{L} = \alpha\left(L_{-1} + L_1\right)$$

action of generators

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$$L_{-1}|h,n\rangle = \sqrt{(n+1)(2h+n)}\,|h,n+1\rangle\,,$$
$$L_1|h,n\rangle = \underbrace{\boxed{\sqrt{n(2h+n-1)}}}_{b_n}\,|h,n-1\rangle\,,$$

SL(2,R) symmetry

[Caputa-Magan-Patramanis]

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b_n

Lanczos coefficients

$$b_n = \alpha\sqrt{n(2h+n-1)} \quad \overset{n\to\infty}{\simeq} \quad \alpha\, n$$

SL(2,R) symmetry

[Caputa-Magan-Patramanis]

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$$L_1\,|h,n\rangle = \sqrt{n(2h+n-1)}\,|h,n-1\rangle\,,$$

b_n

This agrees with SYK calculations using a more direct approach.

Lanczos coefficients

$$b_n = \alpha\sqrt{n(2h+n-1)} \quad \overset{n\to\infty}{\simeq} \quad \alpha\, n$$

[Parker-Cao-Avdoshkin-Scaffidi-Altman]

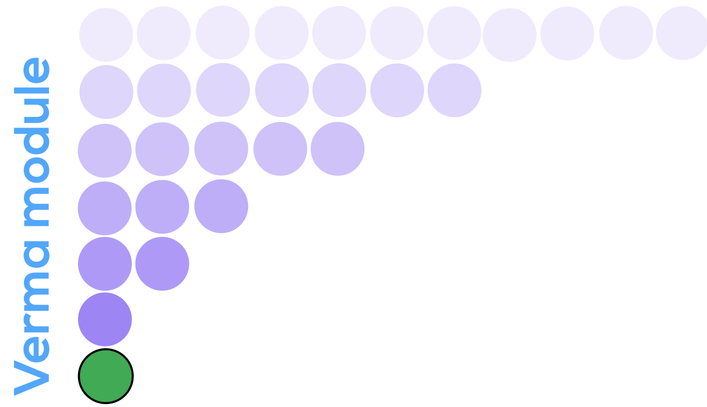
What happens in 2d CFTs?

The situation with 2d CFTs

2d CFTs have an enlarged, infinite dimensional Virasoro symmetry.

There are exact degeneracies from the descendants.

Irrational CFTs (c>1) should thermalize, owing to black hole formation in the gravity dual.



How do operators (e.g. primaries or stress tensor) grow?

Evolution protocol

The standard CFT Hamiltonian is

$$H_{\text{cft}} = \frac{2\pi}{\ell}\left(L_0 + \bar{L}_0 - \frac{c}{24}\right)$$

(Quasi)primaries are eigenstates of the above Hamiltonian.
Therefore, the time evolution is trivial.

We pick a different evolution/driving protocol instead:

$$\mathcal{U}(t) = e^{i\alpha t(L_1 + L_{-1})}$$

Evolution protocol

$$\mathcal{U}(t) = e^{i\alpha t(L_1 + L_{-1})}$$

We can decompose the action using the SL(2,R) algebra

$$e^{(\xi L_{-1} - \bar{\xi} L_1)} = e^{\alpha_- L_{-1}} e^{\alpha_0 L_0} e^{\alpha_+ L_1}$$

$$\xi = re^{i\phi} \qquad \alpha_\pm = \mp e^{\mp i\phi} \tanh(r) \ , \quad \alpha_0 = -2 \log \cosh(r)$$

On the plane, this is a combination of translations, dilatations and special conformal translations.

Evolution of primaries

$$\mathcal{U}(t) = e^{i\alpha t(L_1 + L_{-1})} \qquad \mathcal{L} = \alpha\left(L_{-1} + L_1\right)$$

The action of this unitary on a primary is

$$|\mathcal{O}(t)\rangle \equiv e^{i\alpha(L_{-1}+L_1)t}\mathcal{O}(0)|0\rangle$$

Clearly, this will lead to a superposition of descendants from the primary's Verma module.

Evolution of primaries

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Clearly, this will lead to a superposition of descendants from the primary's Verma module.

We need an orthogonal basis to track the evolution.

$$\langle h|L_2 L_{-1}^2|h\rangle = 6h \neq 0 \qquad \langle h|L_3 L_{-1}^3|h\rangle = 24h \neq 0 \qquad \langle h|L_2 L_1 L_{-1}^3|h\rangle = 10h \neq 0$$

Finite overlaps with non-global descendants.

Can we get an orthogonal basis?

[Polyakov; Zamolodchikov; Besken-SD-Kraus]

The Virasoro generators admit a differential operator realization

$$c = 1 + 24\mu^2 \qquad h = \mu^2 + \lambda^2$$

$$l_0 = h + \sum_{n=1}^{\infty} n u_n \frac{\partial}{\partial u_n} ,$$

$$l_k = \sum_{n=1}^{\infty} n u_n \frac{\partial}{\partial u_{n+k}} - \frac{1}{4} \sum_{n=1}^{k-1} \frac{\partial^2}{\partial u_n \partial u_{k-n}} + (\mu k + i\lambda) \frac{\partial}{\partial u_k} , \qquad k > 0$$

$$l_{-k} = \sum_{n=1}^{\infty} (n+k) u_{n+k} \frac{\partial}{\partial u_n} - \sum_{n=1}^{k-1} n(k-n) u_n u_{k-n} + 2k(\mu k - i\lambda) u_k , \qquad k > 0 .$$

The ingredients can be thought of
as creation and annihilation operators.

$$a_n^\dagger \mapsto u_n \qquad a_n \mapsto \frac{\partial}{\partial u_n} \qquad [a_n, a_n^\dagger] = 1$$

The oscillator formalism

[Polyakov; Zamolodchikov; Besken-SD-Kraus]

An orthogonal basis is furnished by the monomials

$$\Phi_{\{m_i\}}(u) \equiv \frac{u_1^{m_1} u_2^{m_2} \cdots}{\mathcal{N}_{\{m_i\}}} \qquad L_0 \Psi = \left(h \; + \sum_{j=1}^{\infty} j m_j \right) \Psi$$

descendant of $|h\rangle$

$$|\Psi\rangle = (a_1^\dagger)^{m_1} (a_2^\dagger)^{m_2} (a_3^\dagger)^{m_3} \cdots |h\;\rangle$$

Inner product for wavefunctions

$$\big(f(U), g(U)\big) = \int [dU] \overline{f(U)} g(U) \qquad [dU] = \prod_{n=1}^{\infty} d^2 u_n \frac{2n}{\pi} e^{-2n u_n \overline{u}_n}$$

$$\mathcal{N}_{\{m_j\}} = \sqrt{S_{1,m_1} S_{2,m_2} \cdots} = \left[\prod_{j=1}^{\infty} \frac{m_j!}{(2j)^{m_j}} \right]^{1/2};$$

Descendants and integer partitions

The orthogonal descendants have a one-to-one correspondence to integer partitions

$$\Phi_{\{m_i\}}(u) \equiv \frac{u_1^{m_1} u_2^{m_2} \cdots}{\mathcal{N}_{\{m_i\}}} \qquad |1^{m_1} 2^{m_2} \cdots\rangle \mapsto \Phi_{\{m_i\}}(u) \ , \qquad \sum_{j=1}^{N} j m_j = N$$

Equivalently, we can also denote them by Young diagrams

Level 3 descendants

$$|1^3\rangle \mapsto \ydiagram{1,1,1}, \quad |1^1 2^1\rangle \mapsto \ydiagram{2,1}, \quad |3^1\rangle \mapsto \ydiagram{3}$$

$$u_1^3 \qquad\qquad u_1 u_2 \qquad\qquad u_3$$

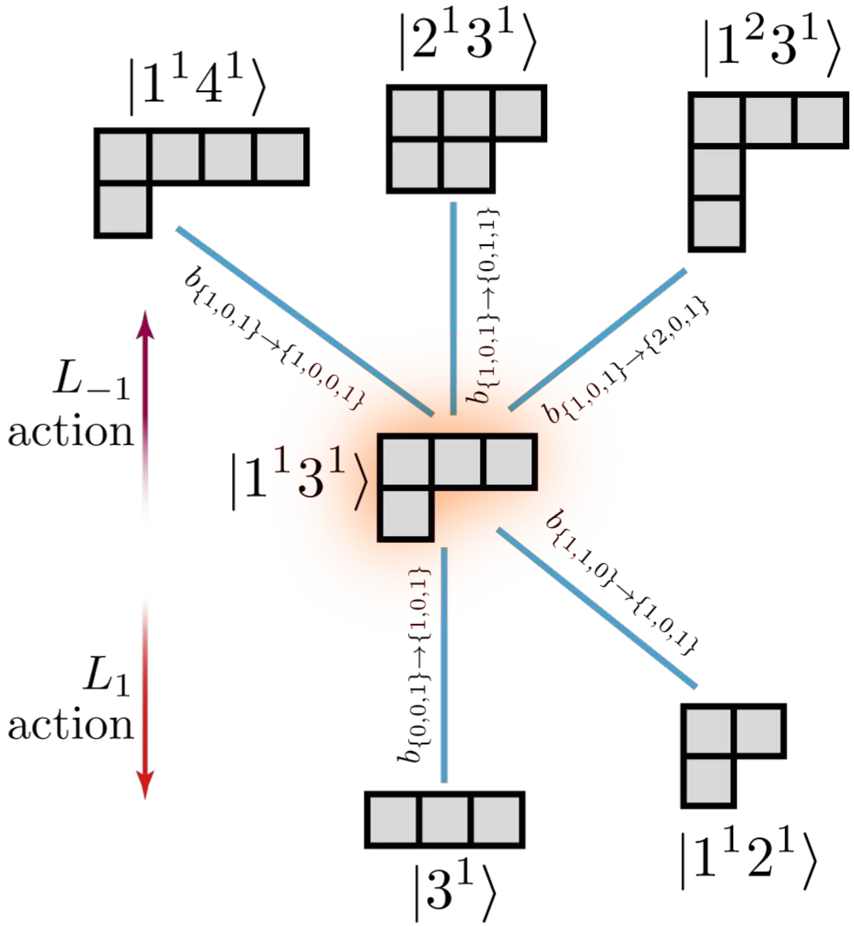
Also, we shall consider irrational c>1 Virasoro CFTs.
There are no null-states.

Lanczos algorithm in 2d CFTs

The action of the Liouvillian

$$|\mathcal{O}(t)\rangle \equiv e^{i\alpha(L_{-1}+L_1)t}\mathcal{O}(0)|0\rangle \longrightarrow \langle u|\mathcal{L}\Phi_{\{m_k\}}\rangle = \alpha(l_{-1}+l_1)\Phi_{\{m_k\}}(u)$$

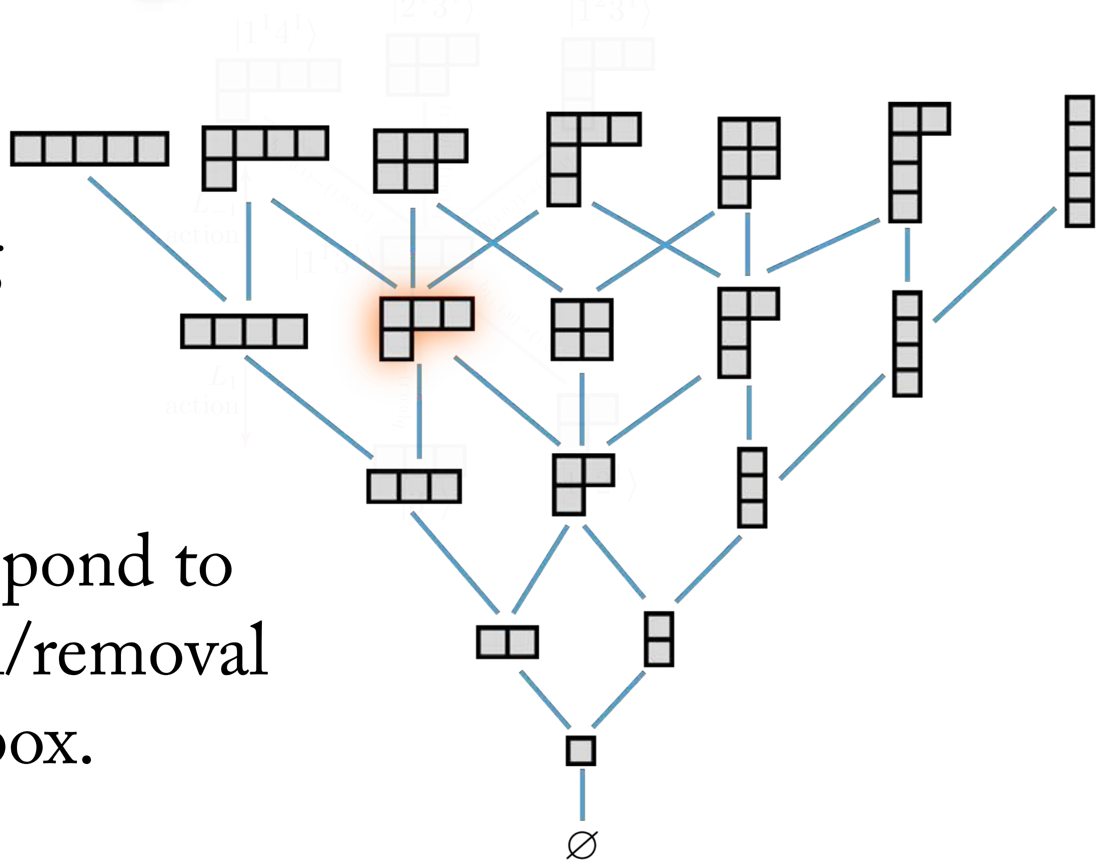
These Virasoro generators perform the job of adding or deleting a single box from a Young diagram.



[Caputa-SD]

The Young's lattice

This is a graph with Young diagrams at the vertices.



The edges correspond to action of addition/removal of a single box.

Weights of the vertices

$$\varphi_{\{m_j\}}(t)$$

wavefunctions

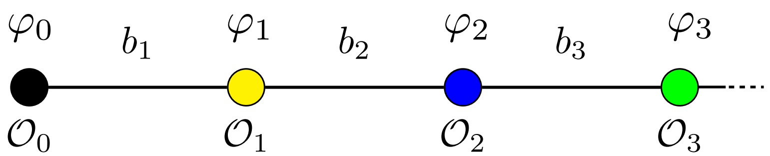
Weights of the edges

$$b_{\{m_k\}\to\{r_j\}}$$

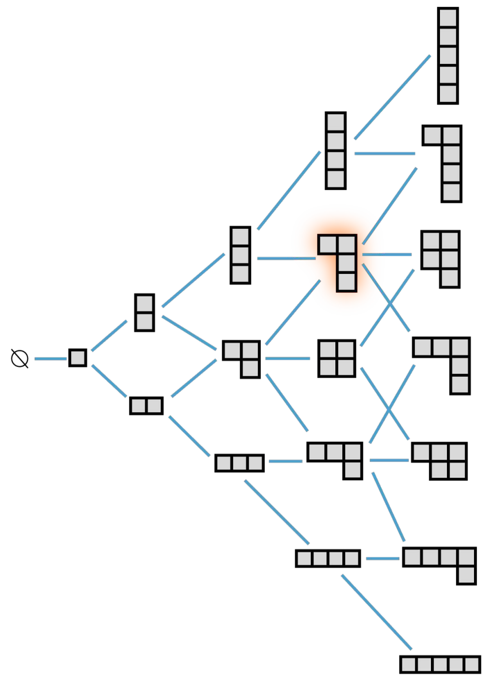
Lanczos coefficients

[Caputa-SD]

The Young's lattice



Standard non-degenerate case



Degenerate Verma module

Weights of the vertices

$$\varphi_{\{m_j\}}(t)$$

wavefunctions

Weights of the edges

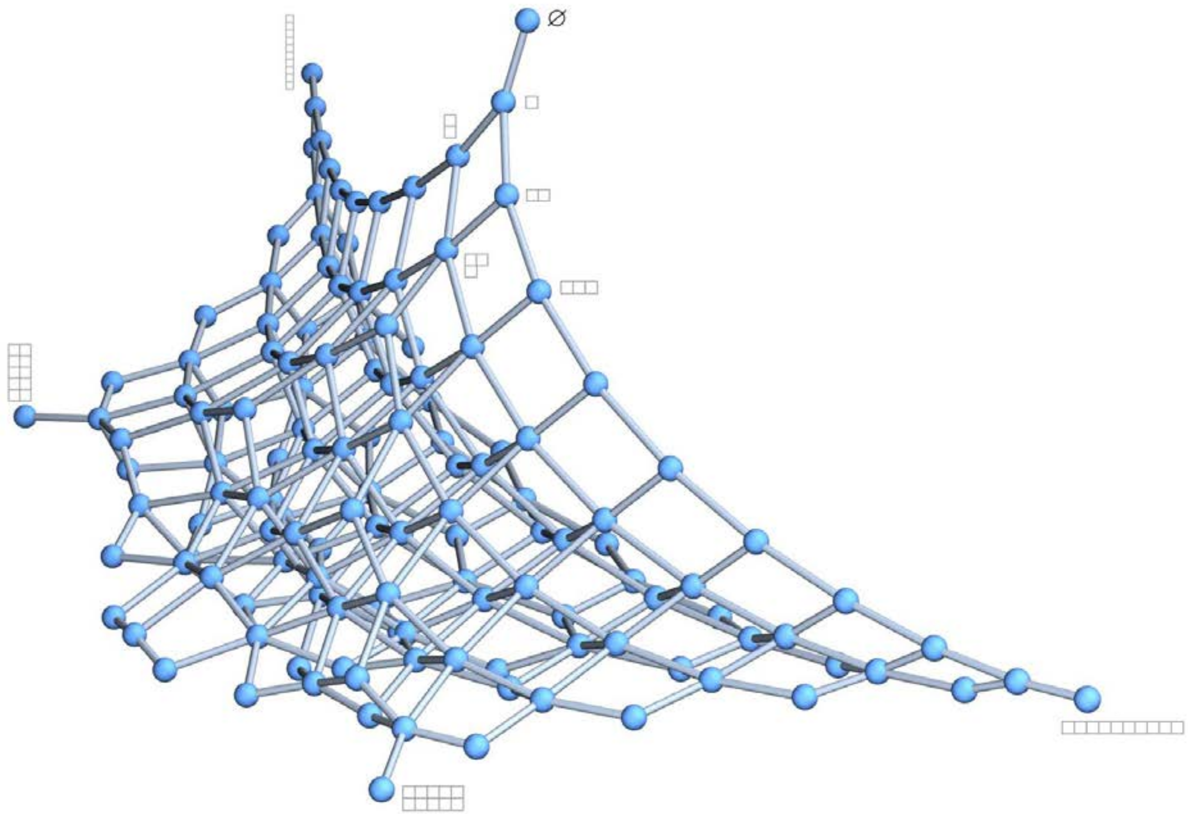
$$b_{\{m_k\}\to\{r_j\}}$$

Lanczos coefficients

[Caputa-SD]

The Young's lattice

[Caputa-SD]



Weights of the vertices

$$\varphi_{\{m_j\}}(t)$$

wavefunctions

Weights of the edges

$$b_{\{m_k\}\to\{r_j\}}$$

Lanczos coefficients

Lanczos coefficients/matrices

[Caputa-SD]

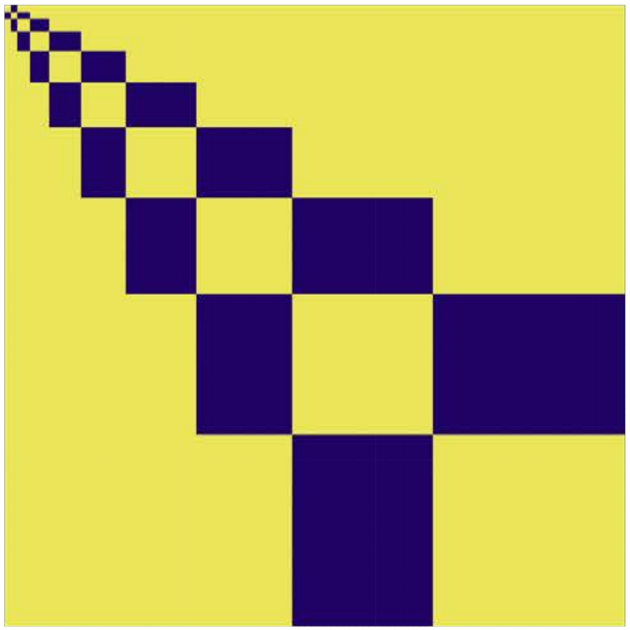
Weights of the edges

$$b_{\{m_k\}\to\{r_j\}}$$

Lanczos coefficients

$$b_{\{m_k\}\to\{r_j\}} = \left(\Phi_{\{m_k\}}(u), \alpha\, l_{-1}\Phi_{\{r_j\}}(u)\right)$$

$$l_{-1} = \sum_{n=1}^{\infty}(n+1)u_{n+1}\frac{\partial}{\partial u_n} + 2(\mu - i\lambda)u_1$$



The Liouvillian matrix acquires a block tridiagonal form.

$$\mathcal{L} = \alpha\,(L_{-1} + L_1)$$

Each block has dimensions

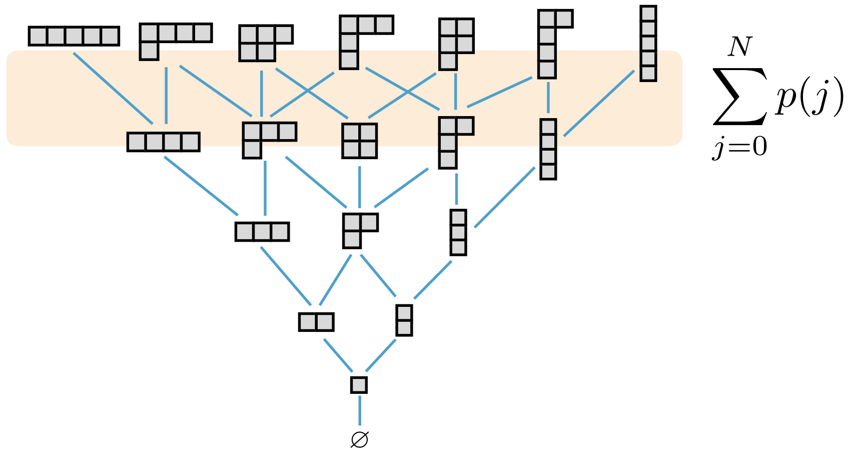
$$p(N)\times p(N\pm 1).$$

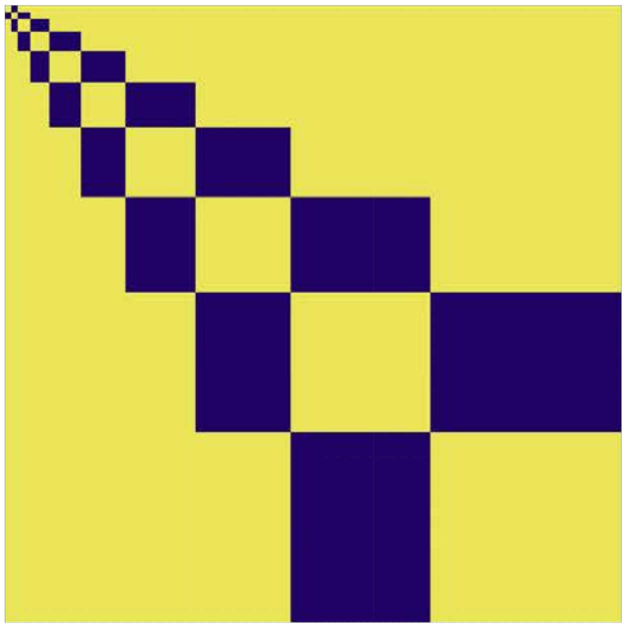
Lanczos coefficients/matrices

Weights of the edges

$$b_{\{m_k\}\to\{r_j\}}$$

Lanczos coefficients





Within the blocks, the non-zero matrix elements correspond to nearest neighbours in the Young's lattice.

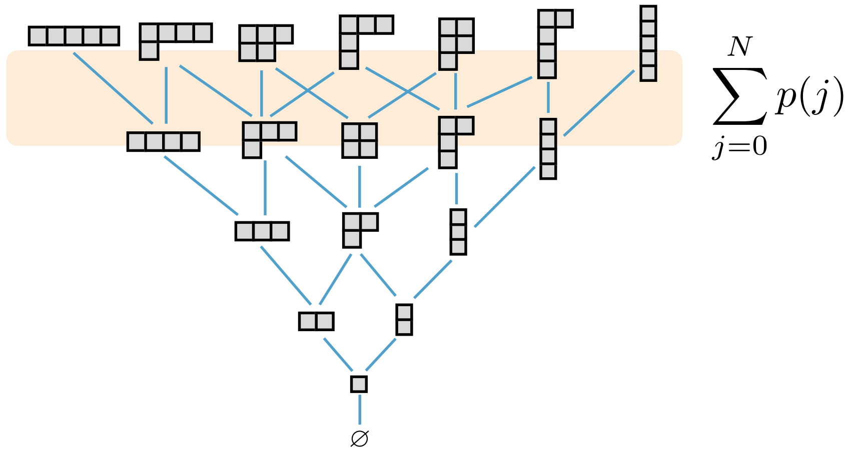
Graph theory lingo: adjacency matrix

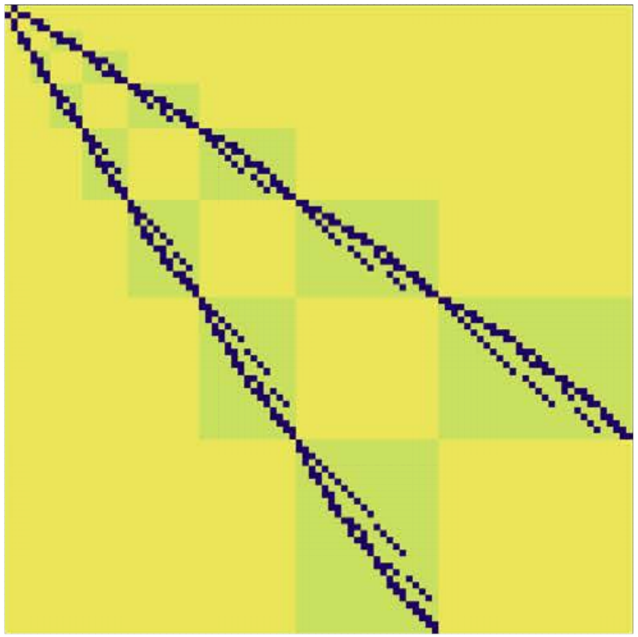
Lanczos coefficients/matrices

Weights of the edges

$$b_{\{m_k\}\to\{r_j\}}$$

Lanczos coefficients





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Graph theory lingo: adjacency matrix

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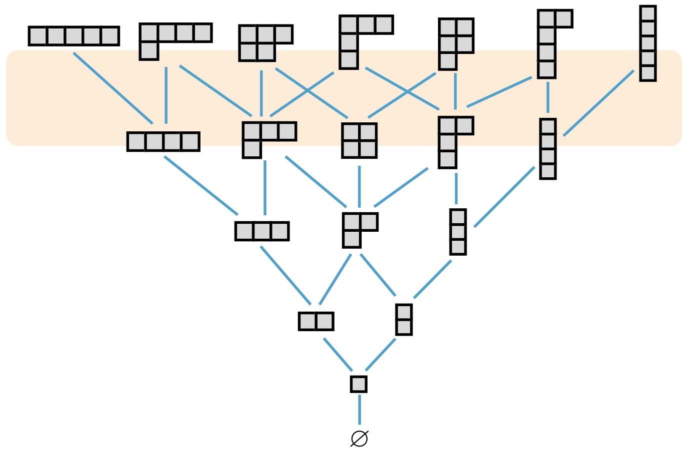
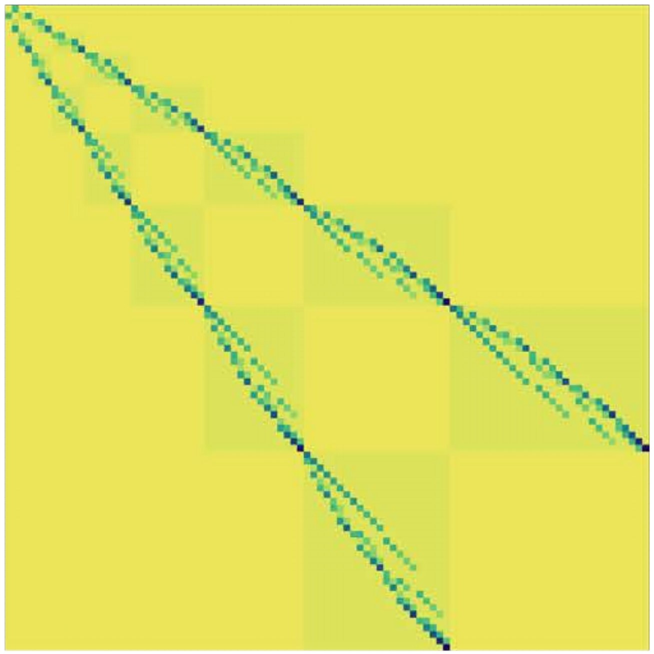


$$\sum_{j=0}^{N} p(j)$$



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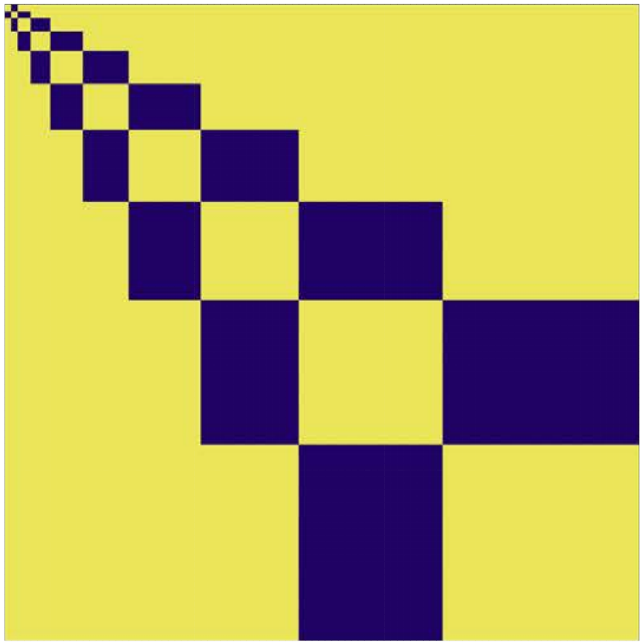
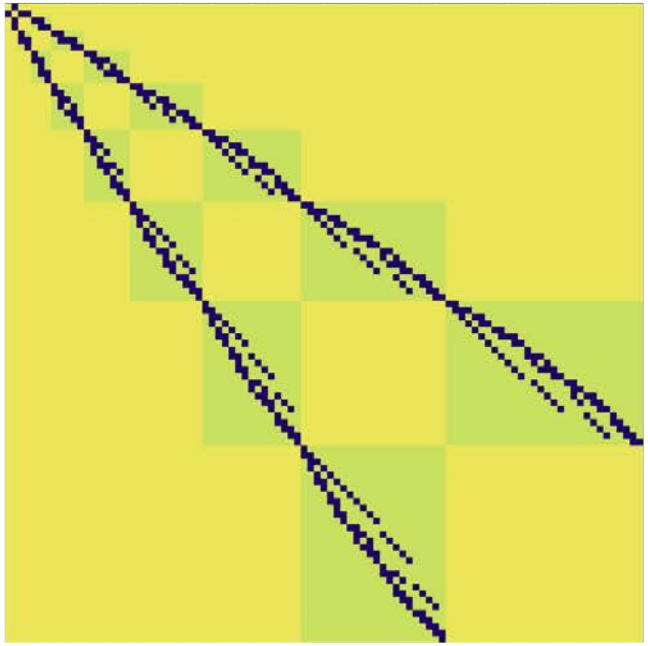
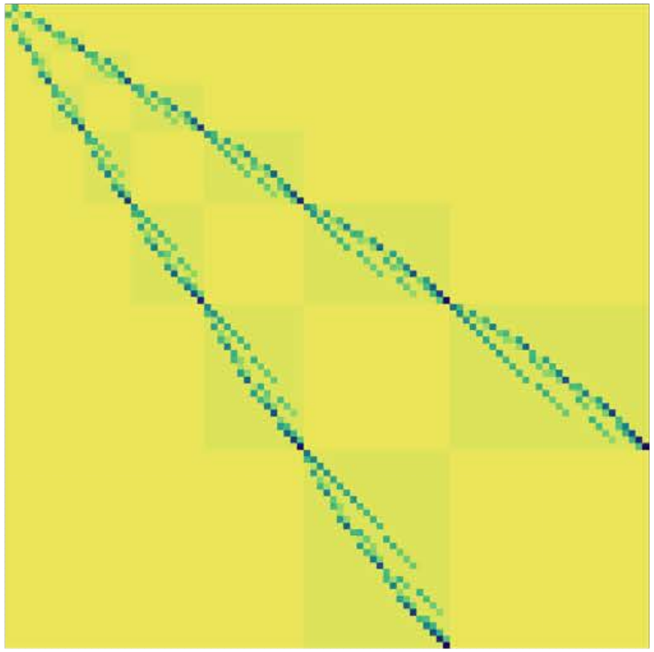
$$l_{-1}=\sum_{n=1}^{\infty}(n+1)u_{n+1}\frac{\partial}{\partial u_n}+2(\mu-i\lambda)u_1$$



tridiagonal blocks



adjacency matrix



actual values

Lanczos coefficients/matrices

[Caputa-SD]

Weights of the edges

$$b_{\{m_k\}\to\{r_j\}}$$

Lanczos coefficients

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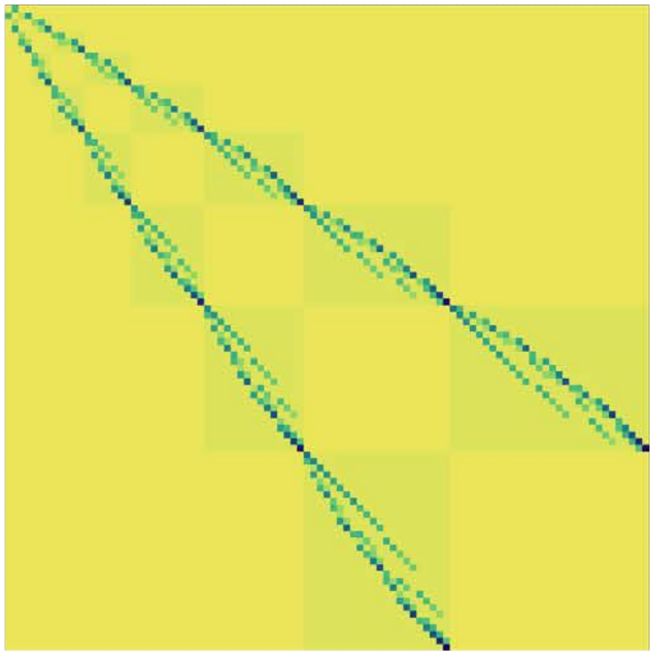
$$l_{-1} = \sum_{n=1}^{\infty}(n+1)u_{n+1}\frac{\partial}{\partial u_n} + 2(\mu - i\lambda)u_1$$

Type-1

$$b^{(1)}_{\{m_j\}\to\{m_1,m_2,\cdots,m_n-1,m_{n+1}+1,\cdots\}} = \alpha\sqrt{n(n+1)m_n(m_{n+1}+1)}$$

Type-2

$$b^{(2)}_{\{m_j\}\to\{m_1+1,m_2,\cdots\}} = \alpha(\mu - i\lambda)\sqrt{2(m_1+1)}$$



actual values

Lanczos coefficients/matrices

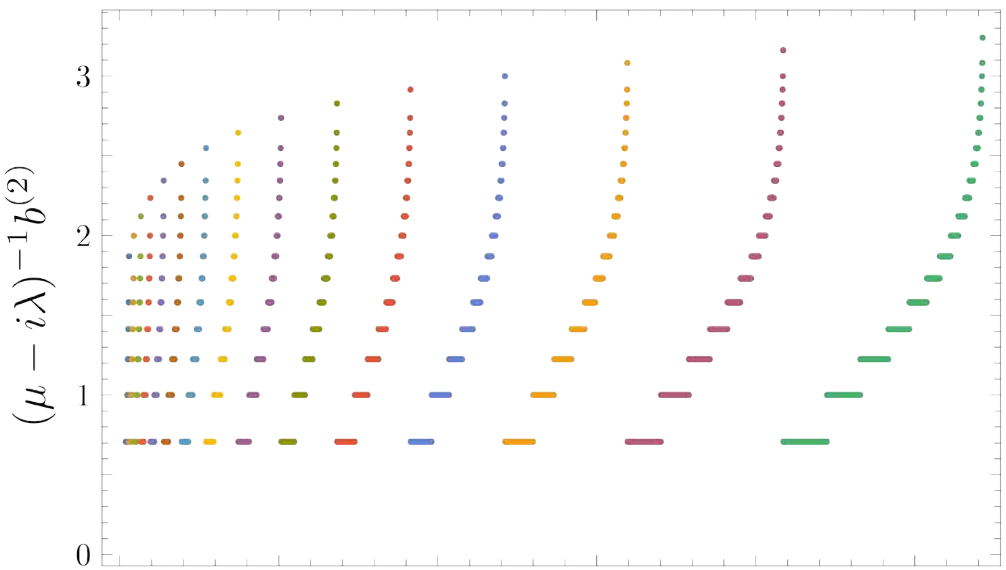
[Caputa-SD]

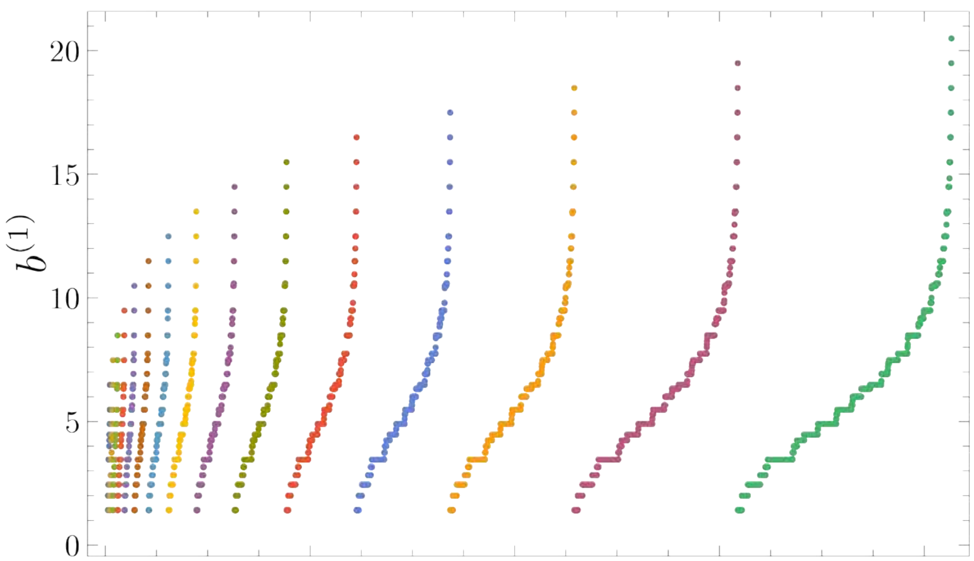
Weights of the edges

$$b_{\{m_k\}\to\{r_j\}}$$

Lanczos coefficients

Type-2 $b^{(2)}_{\dots} = \alpha(\mu - i\lambda)\sqrt{2(m_1+1)}$





Type-1 $b^{(1)}_{\dots} = \alpha\sqrt{n(n+1)m_n(m_{n+1}+1)}$

Each colour denotes transitions between descendant levels N to N+1.

Maximal Lanczos coefficients

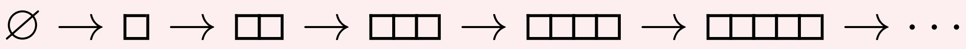
[Caputa-SD]

Weights of the edges

$$b_{\{m_k\}\to\{r_j\}}$$

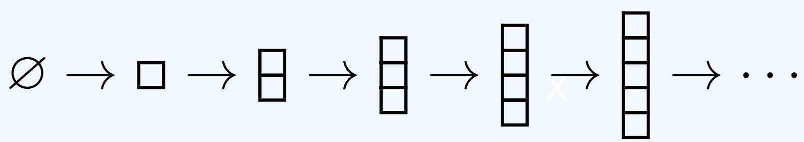
Lanczos coefficients

Type-1 maxima

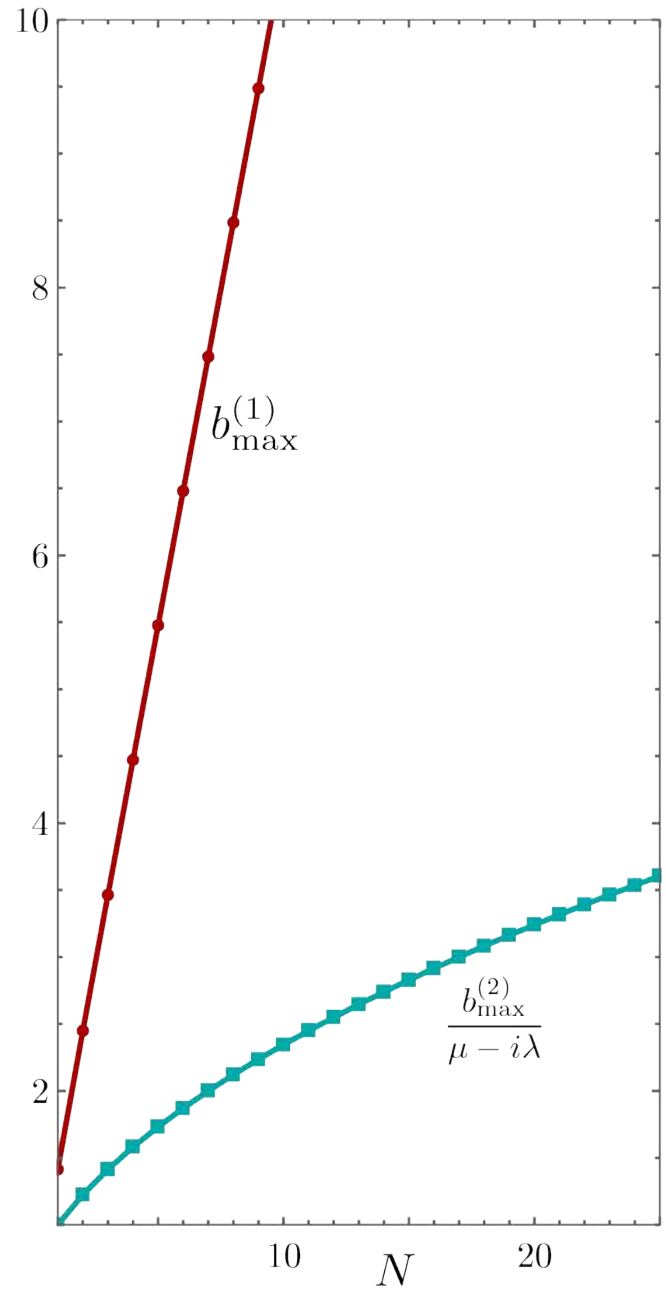


$$b^{(1)}_{\max} = \alpha\sqrt{N(N+1)} \overset{N\to\infty}{\approx} \alpha N$$

Type-2 maxima



$$b^{(2)}_{\max} = \alpha(\mu - i\lambda)\sqrt{2(N+1)} \overset{N\to\infty}{\approx} \alpha(\mu - i\lambda)\sqrt{2N}$$



Maximal Lanczos coefficients

Weights of the edges

$$b_{\{m_k\} \to \{r_j\}}$$

Lanczos coefficients

Type-1 maxima

$$\varnothing \to \square \to \square\square \to \square\square\square \to \square\square\square\square \to \square\square\square\square\square \to \cdots$$

$$b^{(1)}_{\max} = \alpha\sqrt{N(N+1)} \overset{N\to\infty}{\approx} \alpha N$$

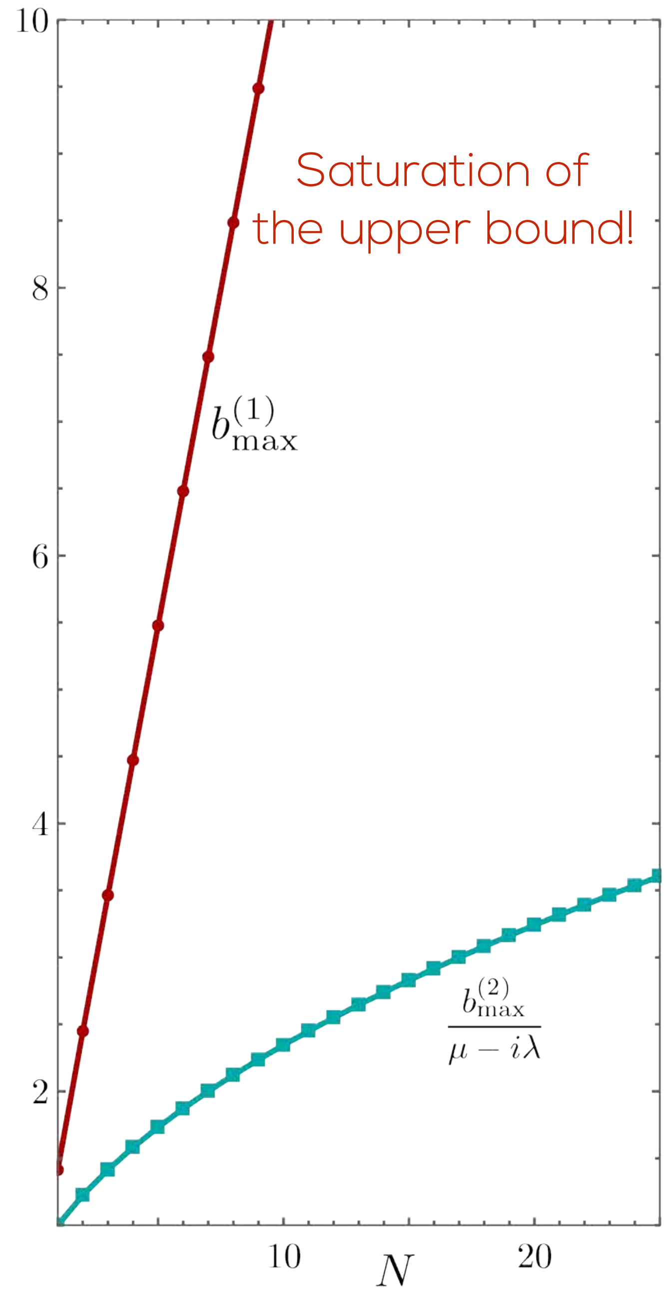
Linear growth

Type-2 maxima

$$\varnothing \to \square \to \begin{smallmatrix}\square\\\square\end{smallmatrix} \to \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix} \to \begin{smallmatrix}\square\\\square\\\square\\\square\end{smallmatrix} \to \begin{smallmatrix}\square\\\square\\\square\\\square\\\square\end{smallmatrix} \to \cdots$$

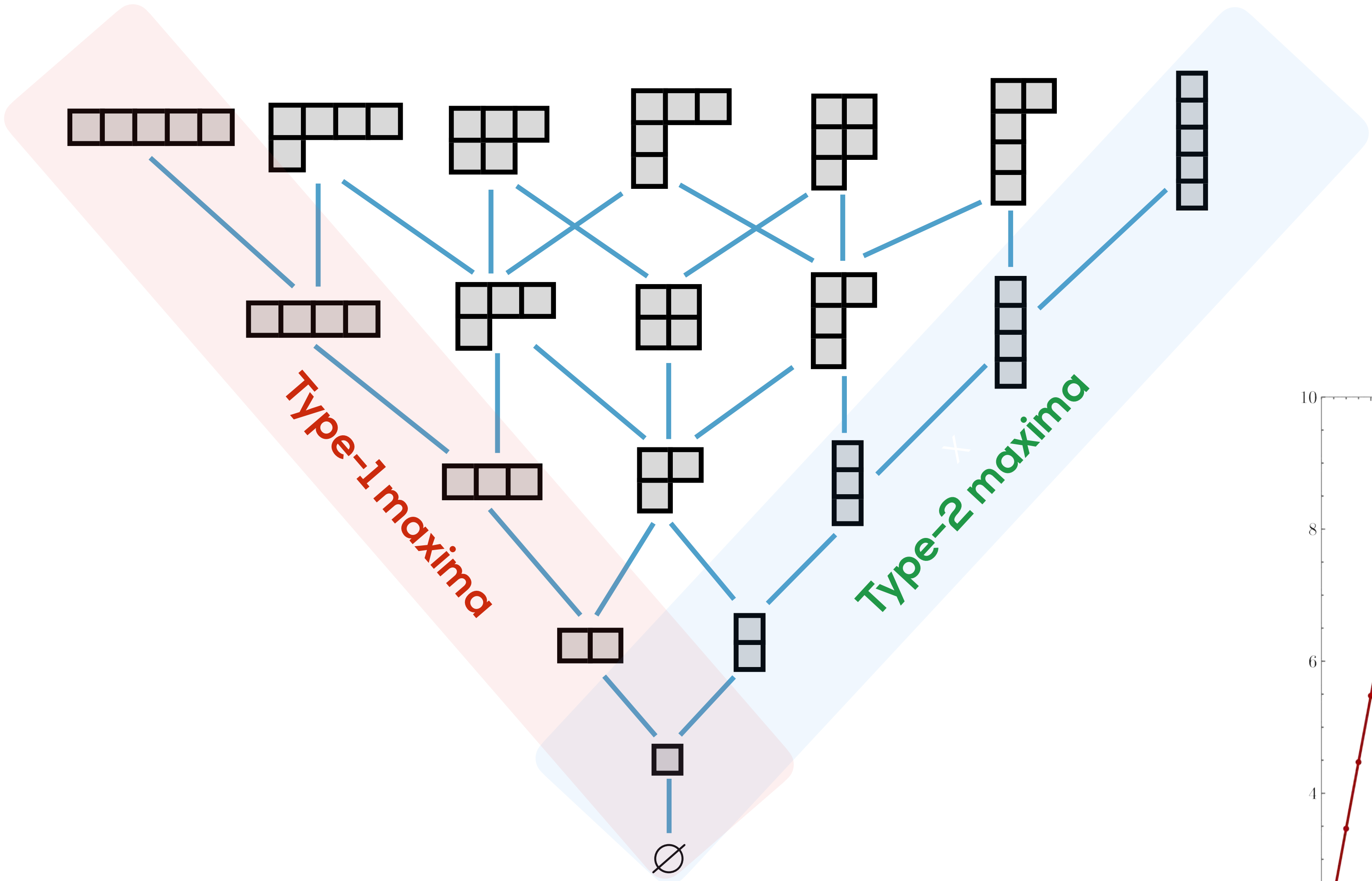
$$b^{(2)}_{\max} = \alpha(\mu - i\lambda)\sqrt{2(N+1)} \overset{N\to\infty}{\approx} \alpha(\mu - i\lambda)\sqrt{2N}$$

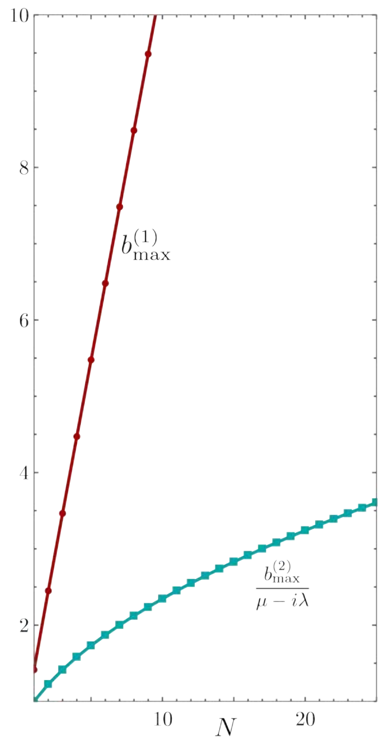
[Caputa-SD]



Maximal Lanczos coefficients

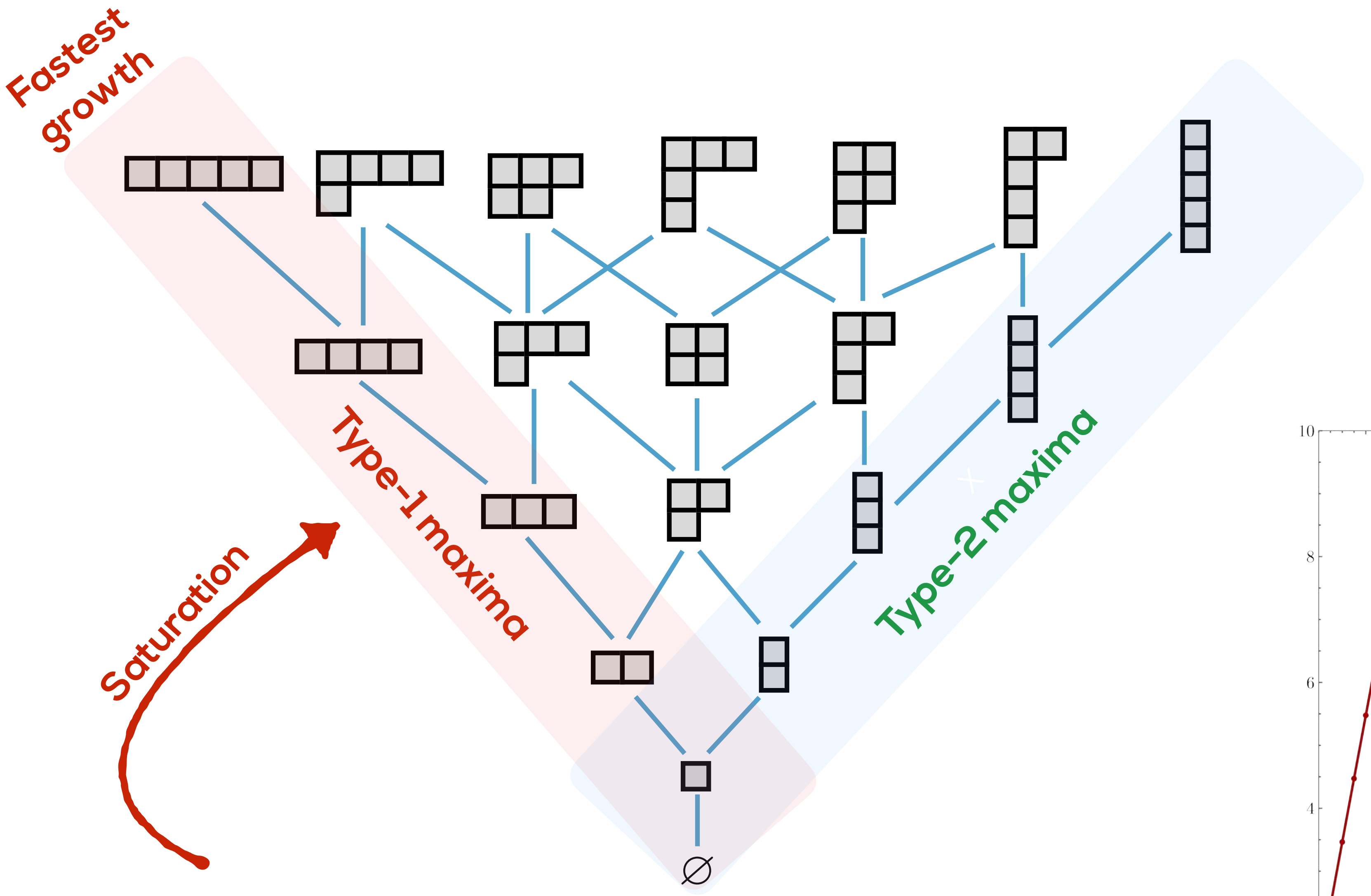
[Caputa-SD]

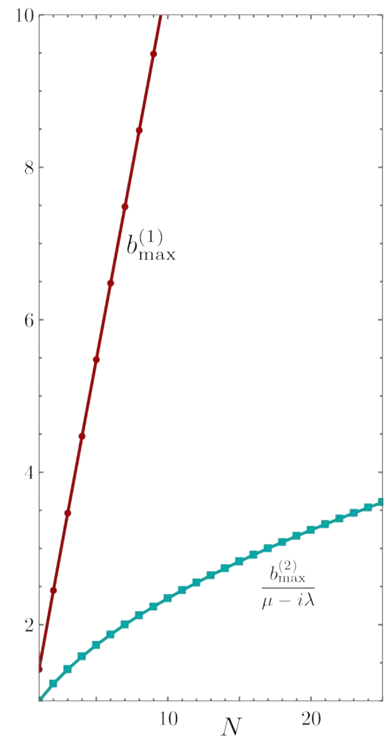




Maximal Lanczos coefficients

[Caputa-SD]





$$b_n \leq \alpha\, n + \mathrm{O}(1)$$

[Parker-Cao-Avdoshkin-Scaffidi-Altman]

Lanczos coefficients/matrices

[Caputa-SD; SD-Michel-Kraus]

Weights of the edges

$$b_{\{m_k\}\to\{r_j\}}$$

Lanczos coefficients

Type-1 maxima

$$\varnothing \to \ydiagram{1} \to \ydiagram{2} \to \ydiagram{3} \to \ydiagram{4} \to \ydiagram{5} \to \cdots$$

$$b^{(1)}_{\text{max}} = \alpha\sqrt{N(N+1)} \overset{N\to\infty}{\approx} \alpha N$$

Type-2 maxima

$$\varnothing \to \ydiagram{1} \to \ydiagram{1,1} \to \ydiagram{1,1,1} \to \ydiagram{1,1,1,1} \to \ydiagram{1,1,1,1,1} \to \cdots$$

$$b^{(2)}_{\text{max}} = \alpha(\mu - i\lambda)\sqrt{2(N+1)} \overset{N\to\infty}{\approx} \alpha(\mu - i\lambda)\sqrt{2N}$$

Lanczos coefficients/matrices

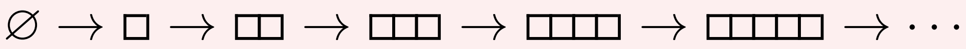
[Caputa-SD; SD-Michel-Kraus]

Weights of the edges

$$b_{\{m_k\}\to\{r_j\}}$$

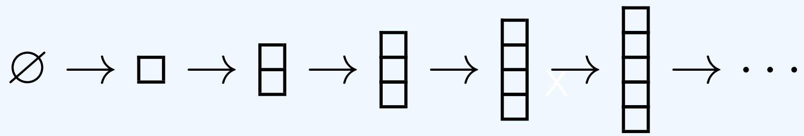
Lanczos coefficients

Type-1 maxima



$$b^{(1)}_{\max} = \alpha\sqrt{N(N+1)} \overset{N\to\infty}{\approx} \alpha N$$

Type-2 maxima



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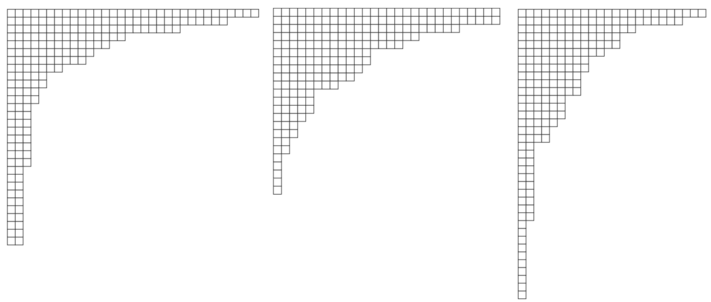
Typical states

$$\overline{m_j} = \frac{1}{e^{\pi/\sqrt{6N}} - 1}$$

$$b^{(1)}_{\text{typ}} \approx \frac{\sqrt{6N}}{\pi}$$

$$b^{(2)}_{\text{typ}} \approx (\mu - i\lambda)\frac{(6N)^{1/4}}{\sqrt{\pi}}$$



Tracking the evolution

The evolved state

[Caputa-SD; Besken-SD-Kraus]

Weights of the vertices

$$\varphi_{\{m_j\}}(t)$$

wavefunctions

$$|\mathcal{O}(t)\rangle \equiv e^{i\alpha(L_{-1}+L_1)t}\mathcal{O}(0)|0\rangle$$

We can evaluate the evolved state in closed form using the oscillator basis.

$$\Psi_{\mathcal{O}}(t) \equiv \langle u|e^{i\alpha t(l_1+l_{-1})}\mathcal{O}(0)|0\rangle = e^{\alpha_0 h}\left[1+\sum_{N=1}^{\infty}(\alpha_-)^N \sum_{\sum j m_j = N} \frac{[2(\mu - i\lambda)]^{\sum m_j}}{\sqrt{T_{1,m_1}T_{2,m_2}\cdots}}\Phi_{\{m_i\}}(u)\right]$$

This allows us to extract the ‘wavefunctions’.

$\alpha_0 = -2\log\cosh(\alpha t)$ $\quad$ $\alpha_- = i\tanh(\alpha t)$ $\quad$ $T_{j,m} = (2j)^m m!$

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Wavefunctions and probabilities

[Caputa-SD; Besken-SD-Kraus]

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The wavefunctions and probabilities can be extracted.

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Probabilities and operator spreading

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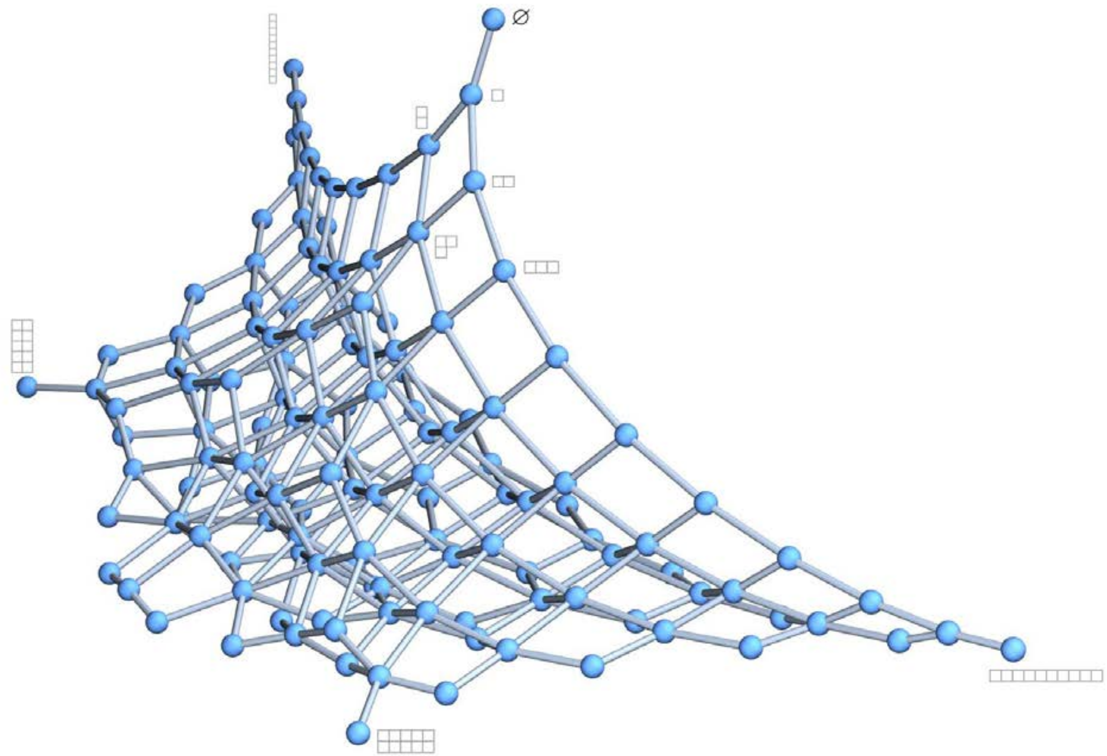
This is the probability of reaching a specific descendant state at time t.



[Caputa-SD]

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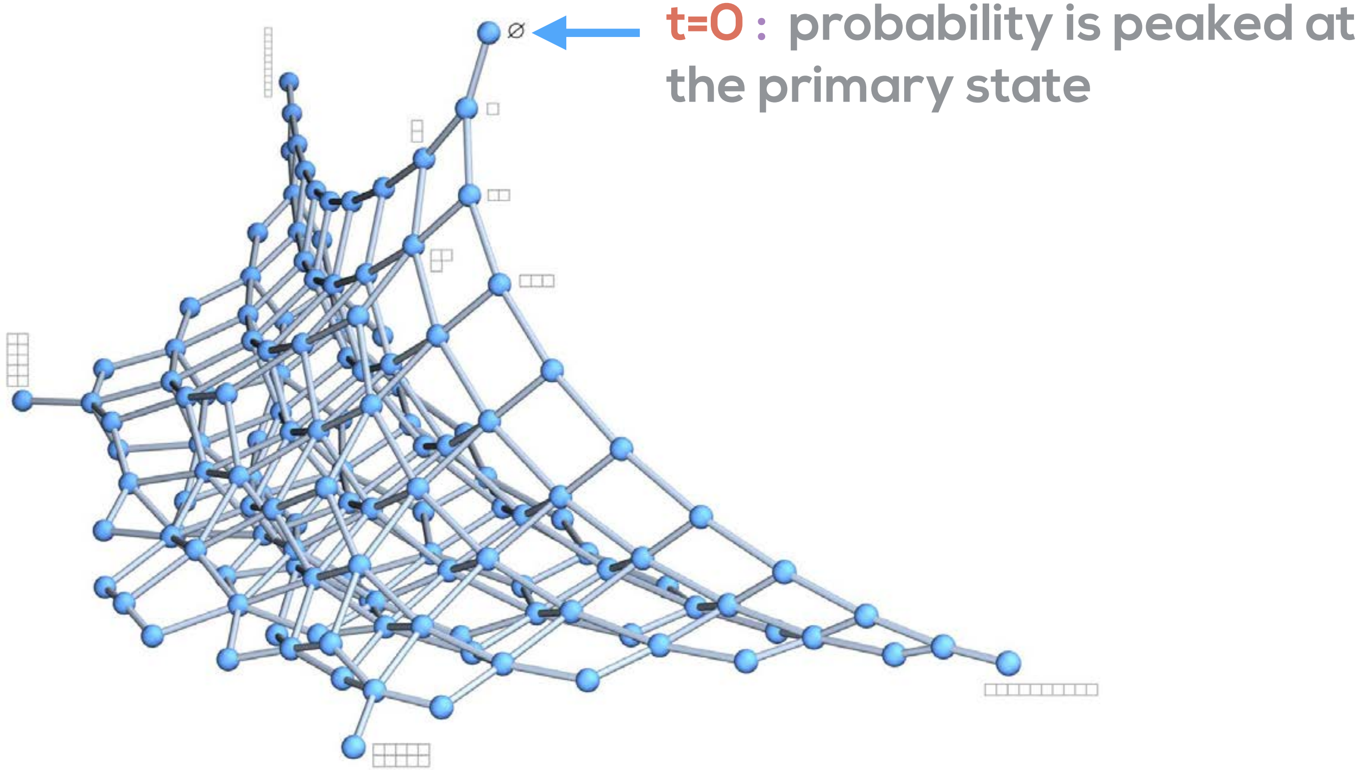

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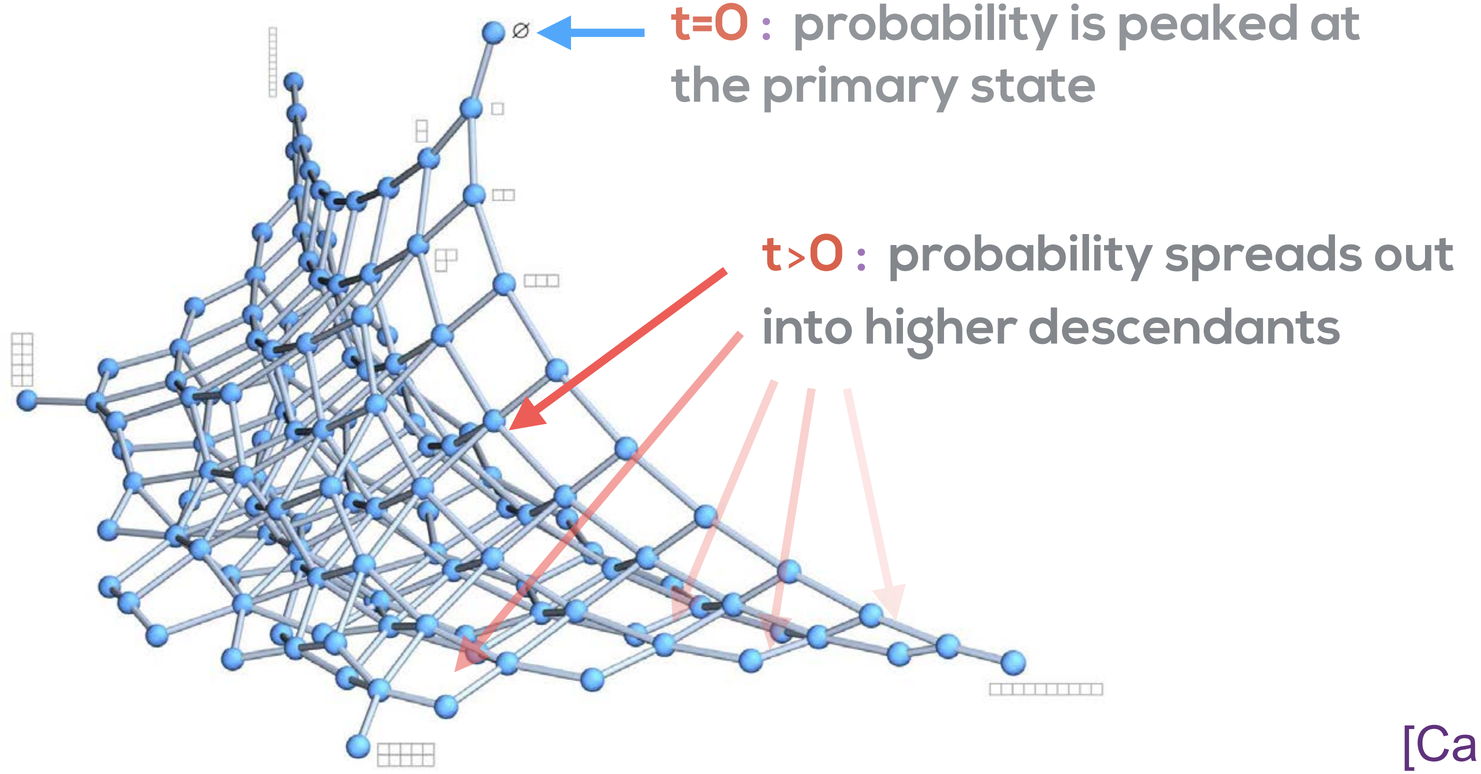

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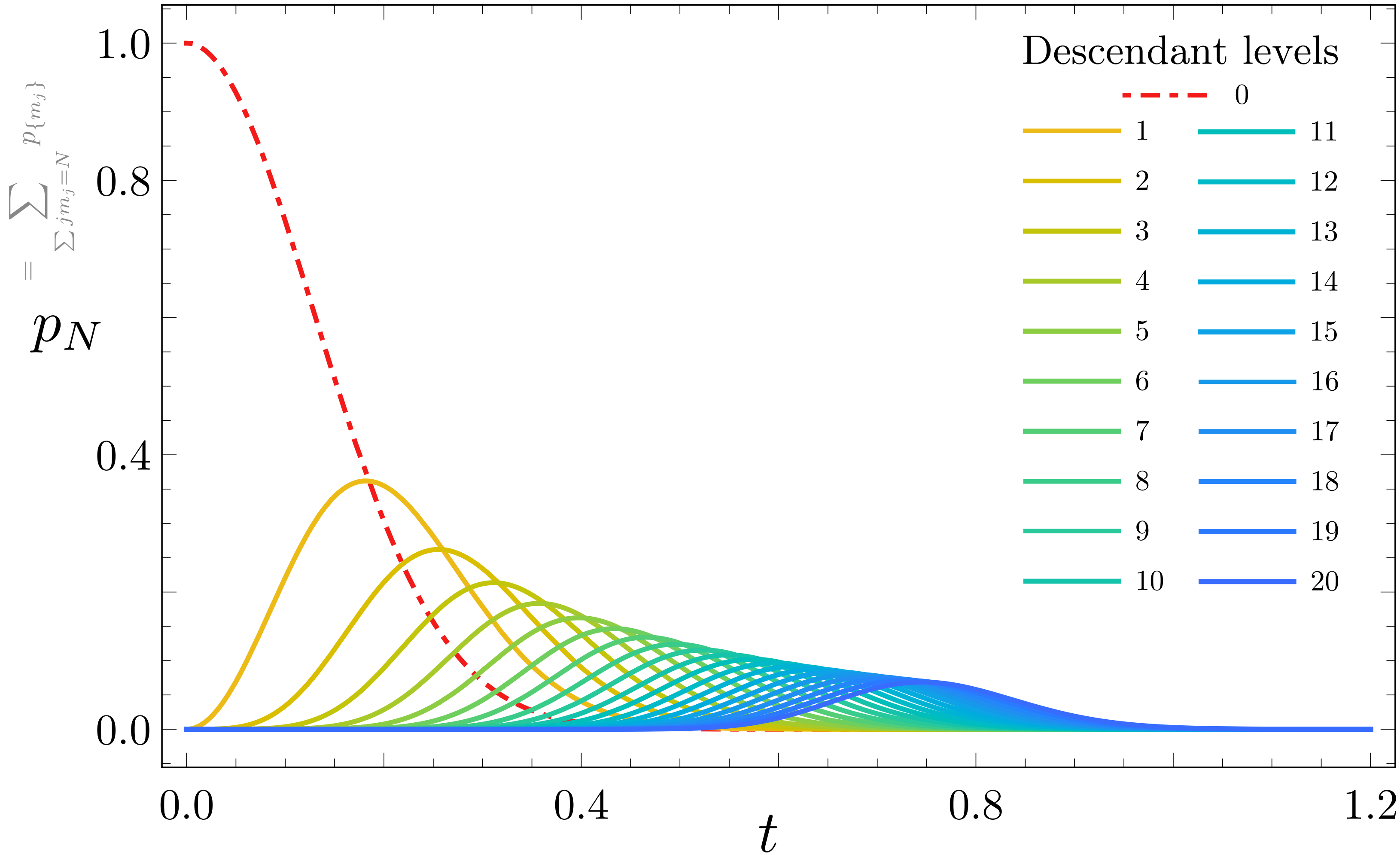

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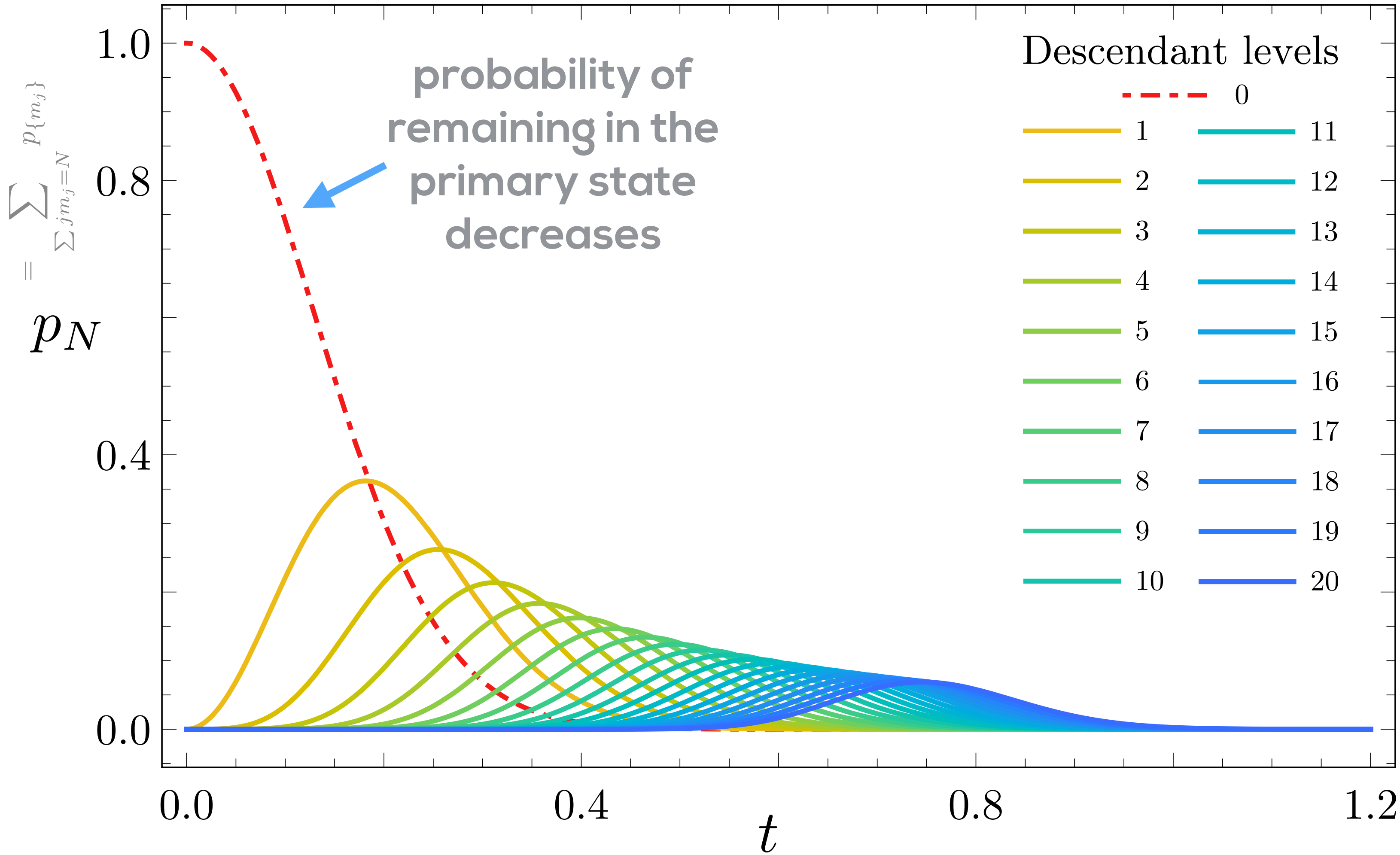

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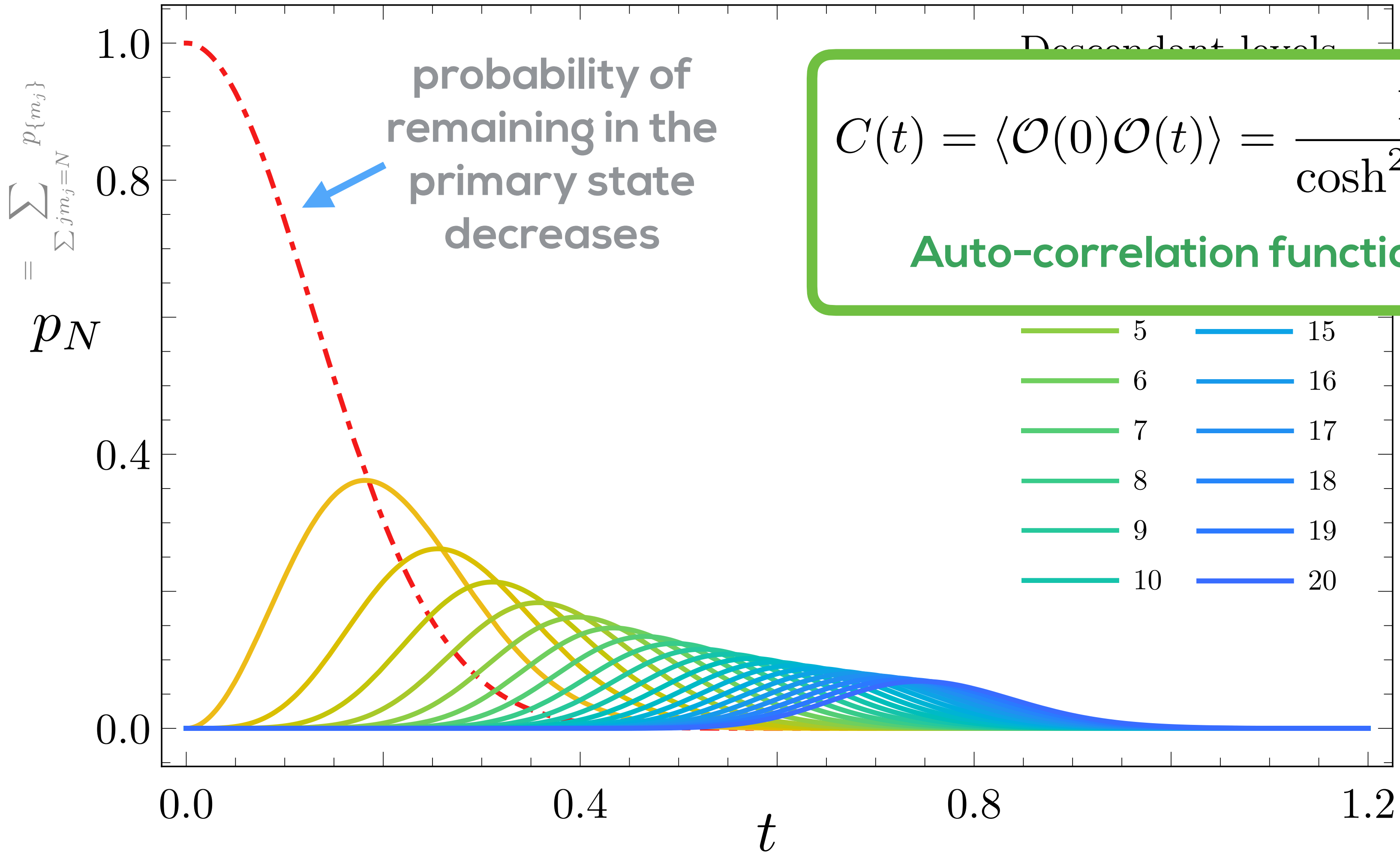

$$C(t) = \langle \mathcal{O}(0)\mathcal{O}(t) \rangle = \frac{1}{\cosh^{2h}(\alpha t)}$$

Auto-correlation function

[Caputa-SD]

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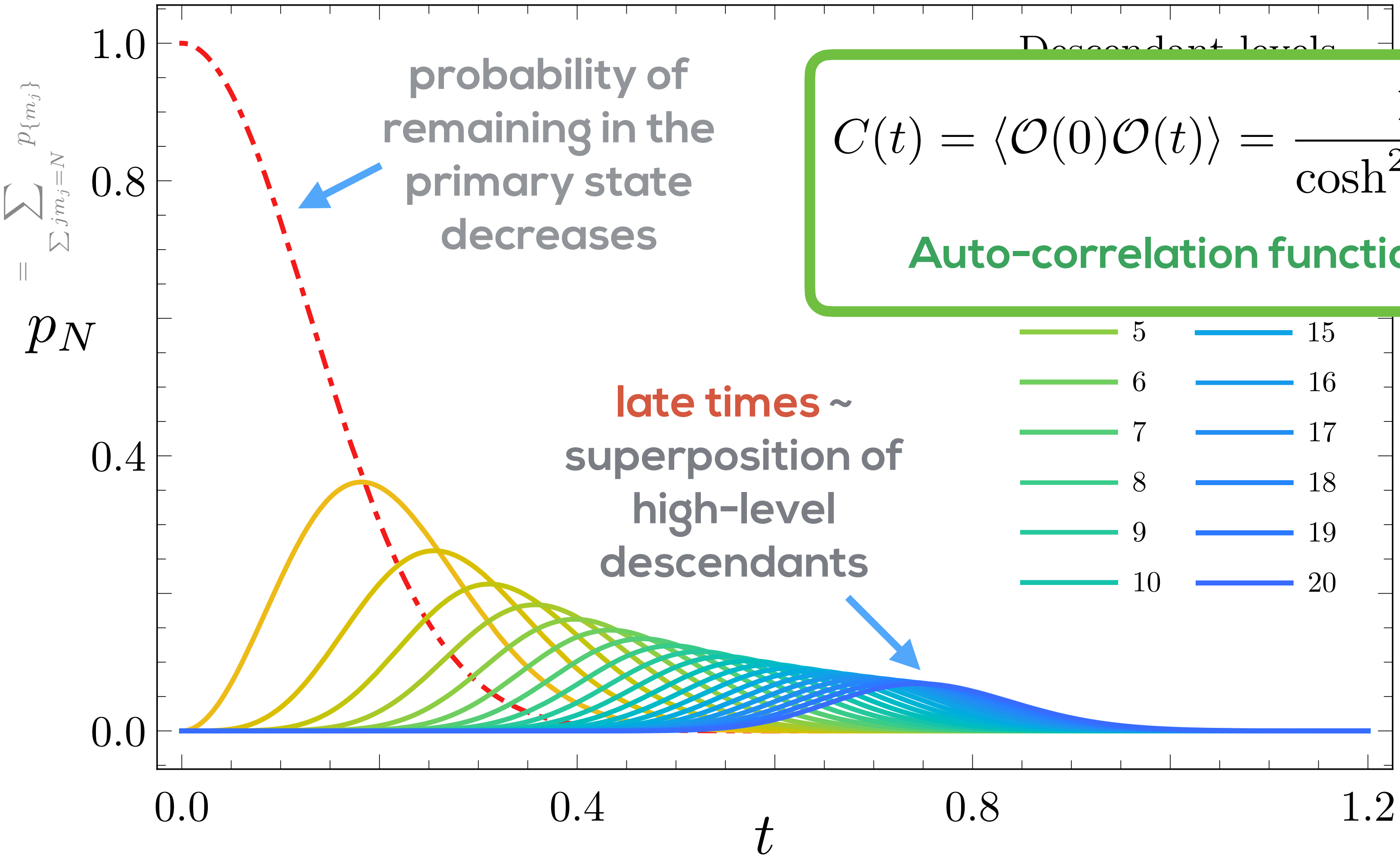

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Krylov complexity

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The probabilities can be used to evaluate
the average descendant level reached at time at time t.

(or avg. layer of the Young's lattice)

This is captured by the Krylov complexity or K-complexity.

$$K_{\mathcal{O}}(t) = \sum_{N=0}^{\infty} N \sum_{\sum j m_j = N} |\varphi_{\{m_j\}}(t)|^2 = 2h \sinh^2(\alpha t)$$

This grows exponentially at late times, demonstrating that the primary operator grows into 'complex' high-level descendants.

[Caputa-SD]

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more fine-grained info

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[Caputa-SD]

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[Caputa-SD]

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$$\delta_{\mathcal{O}}(t)^2 \equiv \frac{\mathrm{Var}[K_{\mathcal{O}}(t)]}{K_{\mathcal{O}}(t)^2} = \frac{\langle \hat{N}^2 \rangle - \langle \hat{N} \rangle^2}{\langle \hat{N} \rangle^2} \, , \qquad \langle \hat{N}^2 \rangle = \sum_{N=0}^{\infty} N^2 \sum_{\sum j m_j = N} |\varphi_{\{m_j\}}(t)|^2$$

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[Caputa-SD]

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Heavy primaries have small fluctuations at late-times.

[Caputa-SD]

Summary of the evolution

Probability of being in a specific descendant state

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A similar analysis can be performed if we started out with the stress tensor as our initial state.

[Caputa-SD]

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Rényi entropy

$$S_K^n = \frac{1}{1-n}\left[\sum_{p=1}^{\infty} \log {}_0F_{n-1}\left(1, 1, \cdots, 1 \middle| \frac{(2h)^n}{p^n} \tanh^{2pn}(\alpha t)\right) - 4nh \log \cosh(\alpha t)\right]$$

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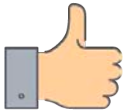

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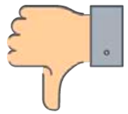

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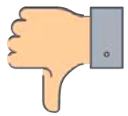

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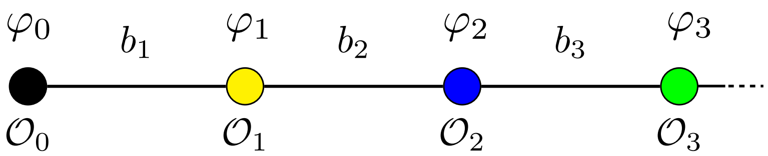
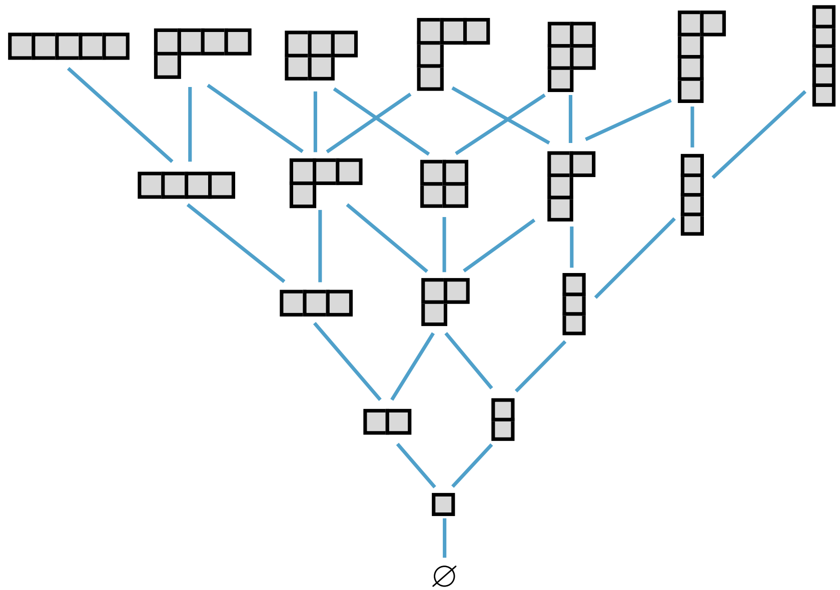
Conclusions & discussions

Takeaways





operator growth in 2d CFTs

$\downarrow$

spreading along the Young's lattice.

We identified the path that saturates the conjectured upper bound on Lanczos coefficients.

Probabilities of reaching specific descendant states were evaluated. The K-complexity shows an exponential growth.

[Caputa-SD]

Some comments

The K-complexity is not a very sensitive probe of operator growth.

It turns out to be the same as the SL(2,R) case
and is, therefore, universal.

The individual Lanczos coefficients and the probabilities
contain much more information than the K-complexity.

The Rényi entropy associated with the probabilities
shows sensitivity to the irrational Virasoro case.

Generalizations

What happens in … ?

Minimal models

Higher-spin CFTs

Symmetric orbifolds

1/2 BPS sector of $\mathcal{N}$=4 SYM

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In the cases above, the states/operators have a Young diagram-like description.

Thank you.