Can we get non-perturbative information from the perturbative coefficients?

$$f(\alpha) = c_0 + c_1\alpha + \ldots + c_n\alpha^n + \ldots$$
$$+e^{-8/\alpha}(d_1 + d_2\alpha + \ldots)$$

Knowing "enough" c-coefficients, can we get the d-s?

Yes!

From perturbative to non-perturbative in the O(4) sigma model

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based on 2011.12254 and 2011.09897

Plan

- Definition of the 2D O(N) models in a magnetic field
- Lagrangean perturbation theory c_1, c_2, c_3

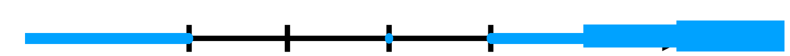
asymptotically free

dynamically generated scale

running coupling α

- Integrable description, Thermodynamic Bethe Ansatz (TBA)
- Expansion of the TBA, perturbative coefficients c_n c_{2000}

- Asymptotic behaviour of c_n, the Borel plane $c_n/n!$
- Analytic structure on the Borel plane



- median resummation and non-perturbative contributions d_n
- resurgence and trans-series $\sum_{m=0}^{\infty} e^{-\frac{2}{\alpha}m} \sum_{n=1}^{\infty} \chi_n^{(m)} \alpha^{n-1}$
- Conclusions

Definition of the model

4 scalar fields in 2D living on the unit sphere $\Phi_1^2 + \ldots + \Phi_4^2 = 1$

magnetic field is coupled the conserved O(N) charge $\mathcal{Q}_{12}$

$$\mathcal{L} = \frac{1}{2\lambda^2}\left\{\partial_\mu \Phi_i \partial^\mu \Phi_i + 2ih(\Phi_1\partial_0\Phi_2 - \Phi_2\partial_0\Phi_1) + h^2(\Phi_3^2 + \Phi_4^2 - 1)\right\}$$

bare coupling

Euclidean Hamiltonian $\mathcal{H} = \mathcal{H}_0 - hQ_{12}$

Perturbation theory $\Phi_1^2 = 1 - \lambda^2(\varphi_2^2 + \varphi_3^2 + \varphi_4^2)$ $\lambda\varphi_i = \Phi_i$

$$e^{-V\mathcal{F}(h)} = \int \mathcal{D}^3[\varphi]\, \mathrm{e}^{-\int d^Dx\, \mathcal{L}(x)}$$

$D = 2 - \epsilon$

dimensional regularisation

Legendre transformation

groundstate energy

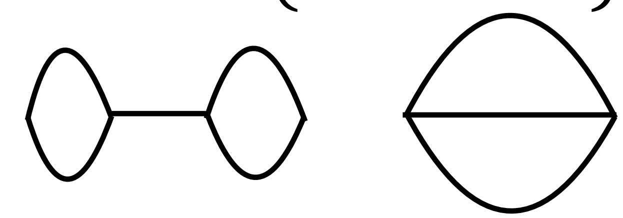
$$\rho = -\partial\mathcal{F}/\partial h, \qquad \epsilon(\rho) = \mathcal{F}(h) - \mathcal{F}(0) + \rho h$$

density

Standard perturbation theory

$$\mathcal{F}(h)-\mathcal{F}(0) = -\frac{h^2}{2\lambda^2}+\frac{N-2}{4\pi}h^{2-\varepsilon}\left\{\frac{1}{\varepsilon}+\frac{\gamma}{2}+\frac{1}{2}\right\}+\lambda^2\frac{N-2}{16\pi^2}h^{2-2\varepsilon}\left\{\frac{1}{\varepsilon}+\gamma+\frac{1}{2}\right\}$$

Bajnok, Balog, Basso, Korchemsky, Palla, *Nucl.Phys.B* 811 (2009) 438 , 0809.4952



renormalized coupling

$$\lambda^2 = (\mu e^{\frac{\gamma}{2}})^{\varepsilon} Z_1 \tilde{g}^2 \qquad \mu\frac{d\tilde{g}}{d\mu} = \beta(\tilde{g}) = -\beta_0\tilde{g}^3 - \beta_1\tilde{g}^5 + \dots$$

$$\mathcal{F}(h)-\mathcal{F}(0) = -\frac{h^2}{2}\left\{\frac{1}{\tilde{g}^2} - 2\beta_0\left(\ln\frac{\mu}{h}+\frac{1}{2}\right) - 2\beta_1\tilde{g}^2\left(\ln\frac{\mu}{h}+\frac{1}{4}\right) + O(\tilde{g}^4)\right\}$$

RG invariant dynamically generated scale

$$\Lambda = \mu e^{-\int^{\tilde{g}}\frac{dg}{\beta(g)}} = \mu e^{-\frac{1}{2\beta_0\tilde{g}^2}}\tilde{g}^{-\beta_1/\beta_0^2}\left[1+\frac{1}{2\beta_0}\left(\frac{\beta_1^2}{\beta_0^2}-\frac{\beta_2}{\beta_0}\right)\tilde{g}^2+\dots\right] \qquad \Delta = \frac{1}{N-2}$$

running coupling

$$\frac{1}{\tilde{\alpha}} + \Delta\ln\tilde{\alpha} = \ln\frac{h}{\Lambda_{\overline{MS}}} \qquad \mathcal{F}(h)-\mathcal{F}(0) = -\beta_0 h^2\left\{\frac{1}{\tilde{\alpha}} - \frac{1}{2} - \frac{\Delta\tilde{\alpha}}{2} + O(\tilde{\alpha}^2)\right\}$$

After Legendre transformation

$$\frac{1}{\alpha} + (\Delta-1)\ln\alpha = \ln\frac{\rho}{2\beta_0\Lambda_{\overline{MS}}} \qquad \epsilon(\rho) = \rho^2\pi\Delta\left\{\alpha + \frac{\alpha^2}{2} + \Delta\frac{\alpha^3}{2} + O(\alpha^4)\right\}$$

Integrable description

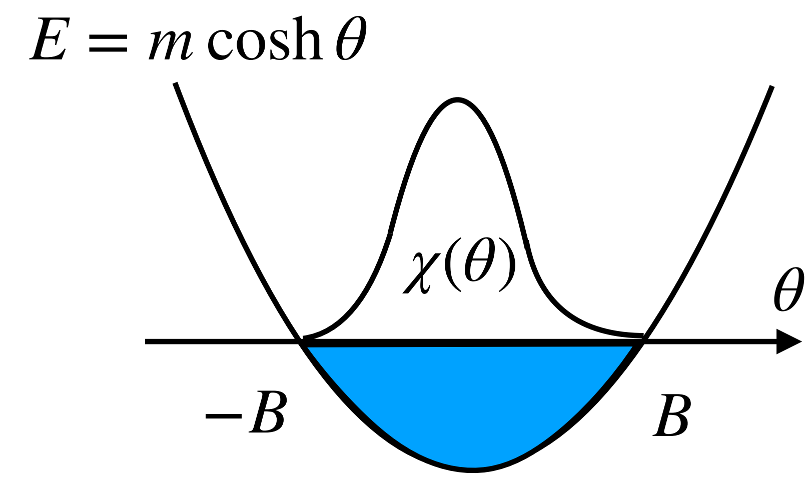
massive particles in the vector representation

$$\mathscr{H} = \mathscr{H}_0 - hQ_{12}$$

particles charged under Q_{12}

condense into the vacuum

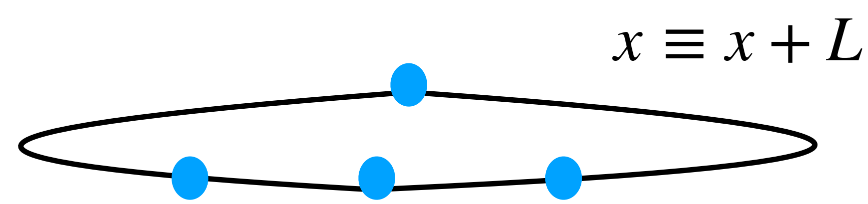
$$E_\pm = m\cosh\theta \pm h$$



multiparticle state on the circle

momentum quantization

$$p = m\sinh\theta$$



$$S(\theta) = -\frac{\Gamma(\frac{1}{2}-\frac{i\theta}{2\pi})\Gamma(\Delta-\frac{i\theta}{2\pi})\Gamma(1+\frac{i\theta}{2\pi})\Gamma(\frac{1}{2}+\Delta+\frac{i\theta}{2\pi})}{\Gamma(\frac{1}{2}+\frac{i\theta}{2\pi})\Gamma(\Delta+\frac{i\theta}{2\pi})\Gamma(1-\frac{i\theta}{2\pi})\Gamma(\frac{1}{2}+\Delta-\frac{i\theta}{2\pi})}$$

$$e^{imL\sinh\theta_j}\prod_k S(\theta_j-\theta_k) = 1$$

Thermodynamic Bethe Ansatz: TBA

$$\chi(\theta) - \int_{-B}^{B}\frac{d\theta'}{2\pi}K(\theta-\theta')\chi(\theta') = m\cosh\theta$$

$$2\pi K(\theta) = -2\pi i\partial_\theta \log S(\theta)$$

density

$$\rho = \int_{-B}^{B}\frac{d\theta}{2\pi}\chi(\theta)$$

ground state energy

$$\epsilon = m\int_{-B}^{B}\frac{d\theta}{2\pi}\cosh\theta\,\chi(\theta)$$

magnetic field

$$h(B) = \partial_\rho\epsilon(\rho) = \frac{\partial\epsilon}{\partial B}/\frac{\partial\rho}{\partial B}$$

Expansion of the TBA

Volin, *Phys.Rev.D* 81 (2010) 105008 • e-Print: 0904.2744

The resolvent

$$R(\theta) = \int_{-B}^{B} d\theta' \, \frac{\chi(\theta')}{\theta - \theta'}$$

its Laplace transform $z = 2(\theta - B)$

$$\hat{R}(s) = \int_{-i\infty+0}^{i\infty+0} \frac{dz}{2\pi i} e^{sz} R(B + z/2)$$

density: residue at ∞

energy:

$$\frac{\epsilon}{m} = \int_{-B}^{B} \cosh\theta \chi(\theta) \frac{d\theta}{2\pi} = \frac{e^B}{4\pi} \hat{R}(1/2)$$

perturbative expansion in 1/B

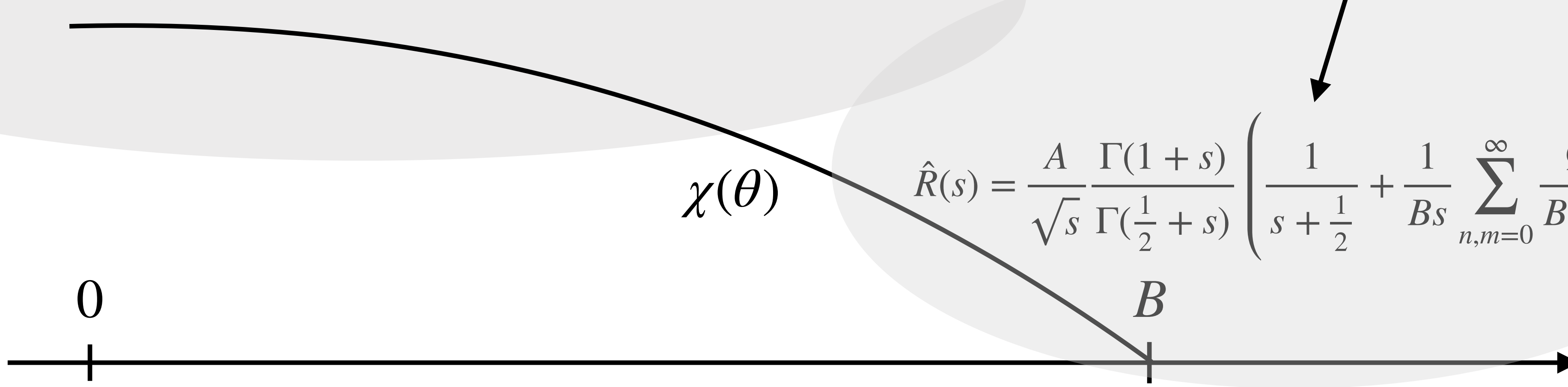
$$\frac{1}{\alpha} + \frac{1}{2} - B - \frac{1}{2}\log B\alpha = \log 2\hat{\rho}$$

Wiener-Hopf

$$A = \frac{m e^B \sqrt{\pi}}{2\sqrt{2}}$$

$$R(\theta) = 2A\sqrt{B} \sum_{n,m=0}^{\infty} \frac{c_{n,m}}{B^{m-n}(\theta^2 - B^2)^{n+1/2}}$$

$$\hat{R}(s) = \frac{A}{\sqrt{s}} \frac{\Gamma(1+s)}{\Gamma(\frac{1}{2}+s)} \left(\frac{1}{s+\frac{1}{2}} + \frac{1}{Bs} \sum_{n,m=0}^{\infty} \frac{Q_{n,m}}{B^{n+m} s^n} \right)$$



$$\rho = A\frac{\sqrt{B}}{\pi}\hat{\rho} = A\frac{\sqrt{B}}{\pi} \sum_{m=0}^{\infty} c_{0,m} B^{-m}$$

$$\epsilon = \frac{m e^B A}{4\sqrt{2\pi}} \left(1 + \sum_{k=0}^{\infty} \epsilon_k B^{-k-1}\right) \qquad \epsilon_k = \sum_{j=0}^{k} 2^{j+1} Q_{j,k-j}$$

Perturbative coefficients

density $$\hat\rho(B) = 1 + \sum_{n=1}^{\infty} \frac{u_n}{B^n}$$

$$u_1 = -\frac{3}{8} + \frac{a}{2} \qquad u_2 = -\frac{15}{128} + \frac{3a}{16} - \frac{a^2}{8} \qquad u_3 = \frac{3\zeta_3}{64} + \frac{a^3}{16} - \frac{9a^2}{64} + \frac{45a}{256} - \frac{105}{1024}$$

$a = \ln 2$ odd zeta functions

energy $$\hat\epsilon(B) = 1 + \sum_{n=1}^{\infty} \frac{\xi_n}{B^n}$$

$$\left\{\frac{1}{4}, \frac{9}{32} - \frac{a}{4}, \frac{a^2}{4} - \frac{9a}{16} + \frac{57}{128}, -\frac{a^3}{4} + \frac{27a^2}{32} - \frac{171a}{128} - \frac{27\zeta_3}{256} + \frac{1875}{2048}, \ldots\right\}$$

Volin, *Phys.Rev.D* 81 (2010) 105008 • e-Print: 0904.2744 — 22 coefficients

Marino,Reis, *JHEP* 04 (2020) 160 • e-Print: 1909.12134 — 44 coefficients

50 coefficients

free energy in the running coupling $$B = \frac{1}{\alpha} + \frac{1}{2} - \ln 2 + \frac{\alpha}{8} + \frac{13 - 18\,\zeta_3}{384}\alpha^3 + \ldots$$

$$f(\alpha) = \frac{\epsilon}{\rho^2} = \frac{\pi}{2}\sum_{n=1}^{\infty} \chi_n \alpha^n \quad = \frac{\pi}{2}\Big(1 + \frac{\alpha}{2} + \frac{\alpha^2}{4} + \frac{10 - 3\zeta_3}{32}\alpha^3 + \chi_5\alpha^4 + \ldots$$

$$\left\{1, \frac{1}{2}, \frac{1}{4}, \frac{5}{16} - \frac{3\zeta_3}{32}, \frac{53}{96} - \frac{9\zeta_3}{64}, -\frac{189\zeta_3}{512} - \frac{405\zeta_5}{2048} + \frac{487}{384}, \ldots\right\}$$

Hasenfratz, Maggiore, Niedermayer, *Phys.Lett.B* 245 (1990) 522

Comparing ordinary perturbation theory in $\frac{h}{\Lambda}$ **to expansion of TBA in** $\frac{h}{m}$

relation between mass and scale can be obtained $m/\Lambda = (8/e)^{\Delta}/\Gamma(1+\Delta)$

Numerical data

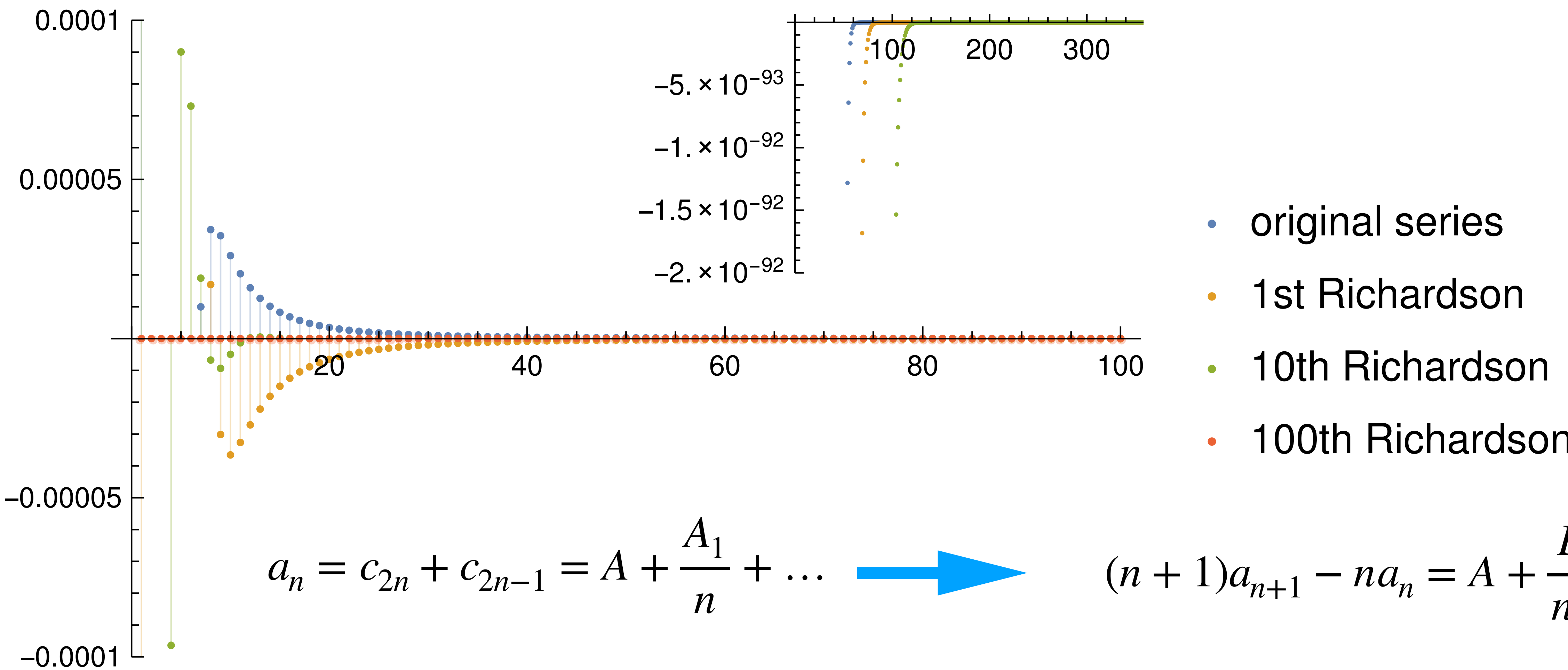
density $$\hat{\rho}(B) = 1 + \sum_{n=1}^{\infty} \frac{u_n}{B^n}$$ 2000 coefficients for 7000 digits

energy $$\hat{\epsilon}(B) = 1 + \sum_{n=1}^{\infty} \frac{\xi_n}{B^n}$$ 2000 coefficients for 7000 digits

free energy $$\hat{f}(\alpha) = \frac{\hat{\epsilon}}{\hat{\rho}^2} = \sum_{n=1}^{\infty} \chi_n \alpha^n$$ 1400 coefficients for 4000 digits

how constant is the asymptotics?

$$c_n = \frac{\chi_{n+2}\, 2^{n+1}}{\Gamma(n+1)} = p^+ + (-1)^n p^- + \frac{1}{n}(\dots)$$



$$a_n = c_{2n} + c_{2n-1} = A + \frac{A_1}{n} + \dots \quad \longrightarrow \quad (n+1)a_{n+1} - n a_n = A + \frac{B}{n^2} + \dots$$

Asymptotic behaviour

perturbative coefficient grow factorially:

how to give meaning to the series?

$$c_n = \frac{\chi_{n+2}\, 2^{n+1}}{\Gamma(n+1)}$$

Borel function

$$B(t) = \sum_{n=1}^{\infty} c_n t^n$$

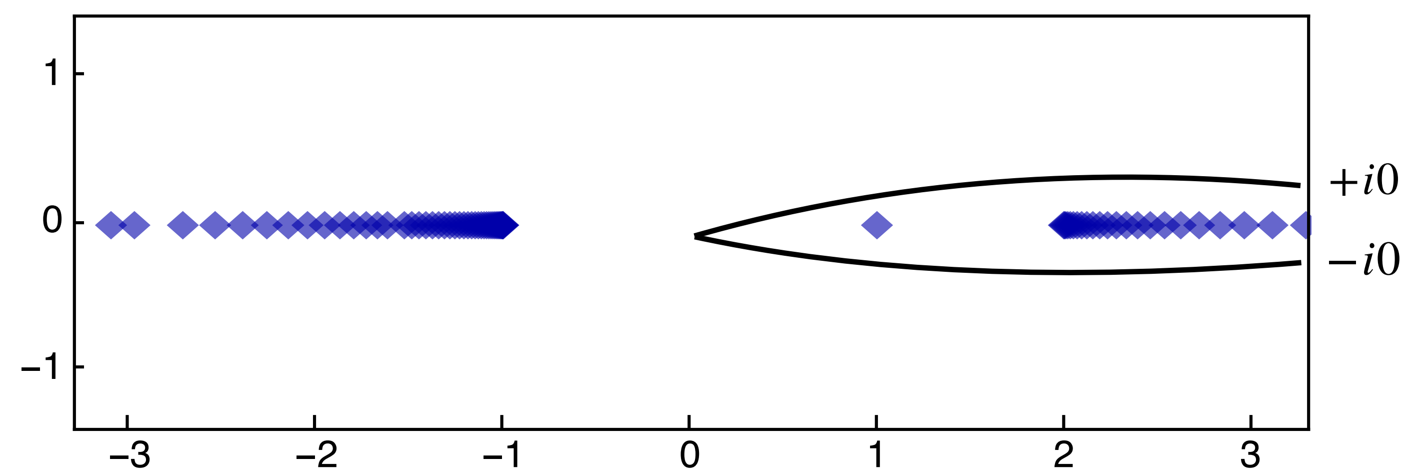
path integral

$$e^{-V\mathcal{F}(h)} = \int \mathcal{D}^3[\varphi]\, \mathrm{e}^{-S[\varphi]} = \int_0^\infty dt e^{-\frac{2t}{\alpha}} \int_{S[\varphi]=\frac{2t}{\alpha}} \mathcal{D}^3[\varphi]\, \mathrm{e}^{-S[\varphi]}$$

convergence radius 1

Pade approximant

$$B(t) \approx \frac{\sum_{i=1}^{n} \beta_i t^i}{1 + \sum_{j=1}^{m} \gamma_j t^j}$$



Ambiguity from the pole

$$\frac{i\pi^2 \alpha \mathrm{res}_1 B(t)}{4} \mathrm{e}^{-\frac{2}{\alpha}}$$

$$f^{(\pm)}(\alpha) = \frac{\pi}{2}\left[\chi_1 \alpha + \chi_2 \alpha^2 + \alpha \int_0^{\infty \pm i0} e^{-\frac{2t}{\alpha}} B(t)\, dt\right]$$

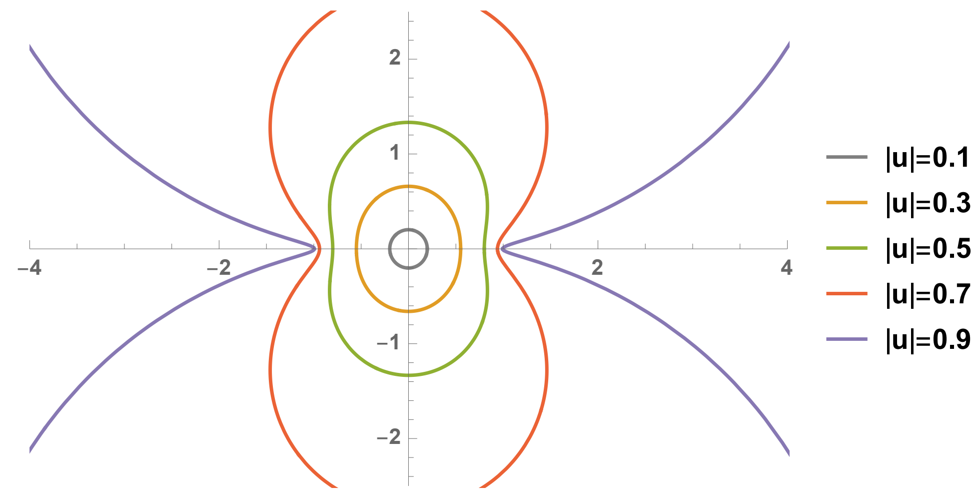
Better approximation ⟶ **Conformal mapping**

Conformal mapping vs numerical solution

Conformal mapping maps the t-plane to the unit u-disk

$$u(t) = \frac{1-\sqrt{1-t^2}}{t}$$

$$\tilde{B}(u) = \sum_{n=1}^{\infty} b_n u^n \qquad \tilde{B}(t) = \sum_{n=1}^{\infty} b_n u(t)^n$$



$$f^{(\pm)}(\alpha) = \frac{\pi}{2}\left[\chi_1\,\alpha + \chi_2\,\alpha^2 + \alpha \int_0^{\infty \pm i0} e^{-\frac{2t}{\alpha}}\, B(t)\, dt\right]$$

Compare to numerical solution in a Chebyshev basis

$$\chi(\theta) = \sum_{j=1}^{(n_c+1)/2} s_j\, T_{2j-2}(\theta/B), \qquad T_n(x) = \cos(n \arccos x)$$

precision=30 digits

$$\mathrm{Im}\left(f^{(+)}(\alpha)\right) = \alpha c_0 \mathrm{e}^{-2/\alpha} + \alpha \mathrm{e}^{-4/\alpha}(c_1 + c_2\alpha + \ldots)$$

$c_0 = 1.70067333(1)$

$c_1 = -1.70067333(1)$

$c_2 = 0.637752(1)$

$c_3 = -0.1727(1)$

$$\mathrm{Re}\left(f^{(+)}(\alpha)\right) = f_{\mathrm{TBA}}(\alpha) + \alpha \mathrm{e}^{-8/\alpha}(d_1 + d_2\alpha + \ldots)$$

$d_1 = -0.9206(1)$

$d_2 = 0.575(3)$

Asymptotic analysis

$\Psi(z) = 1 + \sum_{n=1}^{\infty} s_n/z^n$ $\qquad z = 2B \qquad$ $c_n = s_{n+1}/n!$ $\qquad$ $B(t) = \sum_{n=0}^{\infty} c_n t^n$

Asymptotic large n behaviour

$$c_n = (-1)^n \left(p^- + \frac{p_0^-}{n} + \frac{p_1^-}{n(n-1)} + \dots\right) \qquad + \left(p^+ + \frac{p_0^+}{n} + \frac{p_1^+}{n(n-1)} + \dots\right)$$
$$+ 2^{-n}\left(q^+ + \frac{q_0^+}{n} + \frac{q_1^+}{n(n-1)} + \dots\right)$$

$\log(1+t)\left[-p_0^- + p_1^-(1+t) + \dots\right]$ $\quad + \frac{p^-}{1+t}$ $\qquad \frac{p^+}{1-t}$

$\log(1+t)\Phi(1+t)$

alien derivative $\qquad \Delta_{\pm 1}\Psi(z) = \mp\, i2\pi \left\{ p^{\pm} \pm \sum_{m=0}^{\infty} \frac{(\pm 1)^m p_m^{\pm}}{z^{m+1}} \right\}$

Dorigoni, *Annals Phys.* 409 (2019) 167914 • e-Print: 1411.3585

Aniceto, Basar, Schiappa, *Phys.Rept.* 809 (2019) 1-135 • e-Print: 1802.10441

Asymptotics for $\hat{f}(\alpha)$

numerical fitting 2.8030853547393914280996072422671749861474794385107483226884073330127 5 730867946963527968381041400288 7

$$p^- = -\frac{e}{8\pi}$$ for 150 digits

$$p_0^- = 0$$ for 147 digits

$$p_1^- = \frac{e}{4\pi}$$ for 144 digits

$$p_2^- = \frac{\mathrm{e}}{4\pi}\left(-\frac{1}{2} - \frac{3}{4}\zeta_3\right)$$

http://wayback.cecm.sfu.ca/projects/EZFace/

$\Delta_{-1}\hat{f}$

$$p^+ = \frac{8}{e\pi}$$

$$p_0^+ = 0$$

$$p_1^+ = 0$$

$\Delta_1\hat{f}$

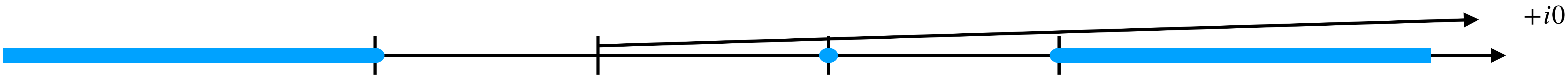
$$q^- = \frac{16}{e^2\pi}$$ for 80 digits

$$q_0^- = \frac{16}{e^2\pi}\left(-\frac{3}{4}\right)$$

$$q_0^- = \frac{16}{e^2\pi}\left(\frac{13}{32}\right)$$

$$q_0^- = \frac{16}{e^2\pi}\left(-\frac{99}{256} + \frac{3}{8}\zeta_3\right)$$

$\Delta_2\hat{f}$



imaginary ambiguity

$$f^{(+)}(\alpha) = \frac{\pi}{2}\left[\alpha \int_0^{\infty+i0} e^{-\frac{2t}{\alpha}} B(t)\, dt\right]$$

$$\Im\mathrm{m}(f^{(+)}(\alpha)) = \frac{4\pi}{\mathrm{e}^2}\alpha \mathrm{e}^{-2/\alpha} + \alpha \mathrm{e}^{-4/\alpha}\left(-\frac{4\pi}{\mathrm{e}^2} + \frac{3\pi}{2\mathrm{e}^2}\alpha - \frac{13\pi}{32\mathrm{e}^2}\alpha^2 + \ldots\right)$$

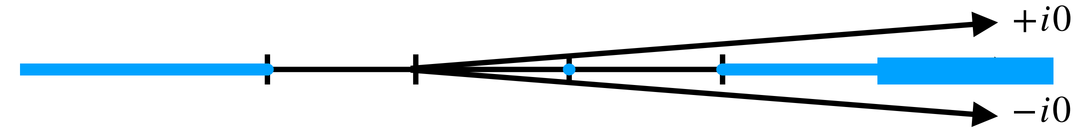
$c_0 = 1.70067333$ $c_1 = -1.70067333(1)$ $c_2 = 0.637752(1)$ $c_3 = -0.1727(1)$

real ambiguity???

Median resummation and Stokes automorphism

$x = \frac{2}{\alpha}$

$$S_\pm(f) = f^{(\pm)} = \chi_1 + \alpha\chi_2 + \int_0^{\infty \pm i0} e^{-tx} B(t) dt$$



$$S_+(f) - S_-(f) = -S_+\left(e^{-x}\Delta_1 f + e^{-2x}\Delta_2 f + \ldots + \frac{e^{-2x}}{2}\Delta_1^2 f + \ldots\right)$$

cut of the cut of the cut…

$$S_+(f) = S_-(\mathfrak{S}f) \quad ; \quad S_-(f) = S_+(\mathfrak{S}^{-1}f)$$

$$\mathfrak{S} = \exp\left\{-\sum_{n=1}^{\infty} e^{-nx}\Delta_n\right\}$$

Median resummation

$$S_{\rm med}(f) = S_-(\mathfrak{S}^{\frac{1}{2}}f) = S_+(\mathfrak{S}^{-\frac{1}{2}}f) = S_+(e^{\frac{1}{2}\sum e^{-nx}\Delta_n}f)$$

$$S_{\rm med}(f) = S_+\left(f + \frac{e^{-x}}{2}\Delta_1 f + \frac{e^{-2x}}{2}\Delta_2 f + \ldots + \frac{e^{-4x}}{8}\Delta_1\Delta_3 f + \frac{e^{-4x}}{8}\Delta_2^2 f + \ldots\right)$$

$$\Delta_1 f = -\frac{16i}{\mathrm{e}^2} \qquad \Delta_2 f = \frac{16i}{\mathrm{e}^2}\left(1 - \frac{3}{4x} + \frac{13}{32x^2} - \left(\frac{99}{256} - \frac{3}{8}\zeta_3\right)\frac{1}{x^3} + \ldots\right)$$

We need $\Delta_2\Delta_2 f$

need the asymptotics of $\Delta_2 f$

$$S_+(\Delta_2 f) - S_-(\Delta_2 f) = -S_+(e^{-2x}\Delta_2\Delta_2 f) + \ldots$$

Spoiler $S_{\rm med}(f) = \mathfrak{Re}(S_+(f)) + \frac{32}{\mathrm{e}^4}\mathrm{e}^{-8/\alpha}(1 - \frac{5\alpha}{8} + \ldots)$ $\qquad d_1 = 0.58607 \qquad \frac{d_2}{d_1} = -0.6246$

Resurgence in 1/B

Alien derivatives from asymptotics

$\hat{\rho}(B)$

$\Delta_{-1}\hat{\rho} = i\hat{\epsilon}\hat{\rho},$ $\quad \Delta_1\hat{\rho} = 0,$ $\quad \Delta_2\hat{\rho} = iR/2,$ $\quad R = 1 + \sum_{n=1}^{\infty} r_n/z^n\,.$

$\Delta_{-1}\hat{\epsilon} = i\hat{\epsilon}^2,$ $\quad \Delta_1\hat{\epsilon} = -4i\,.$ $\quad \Delta_2\hat{\epsilon} = 2iE,$ $\quad E = 1 + \sum_{n=1}^{\infty} e_n/z^n$

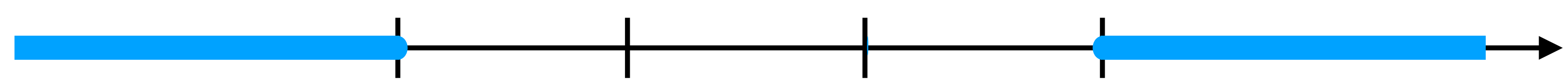
$\hat{\epsilon}(B)$

asympotics of the asymptotics: analytic structure of the alien derivatives

$R(B)$ $\quad r_1 = 1/2 + a$ $\quad r_2 = -a/2 - a^2/2$ $\quad r_3 = \frac{21}{64} + \frac{3}{4}a^2 + \frac{a^3}{2} + \frac{3}{8}\zeta_3$

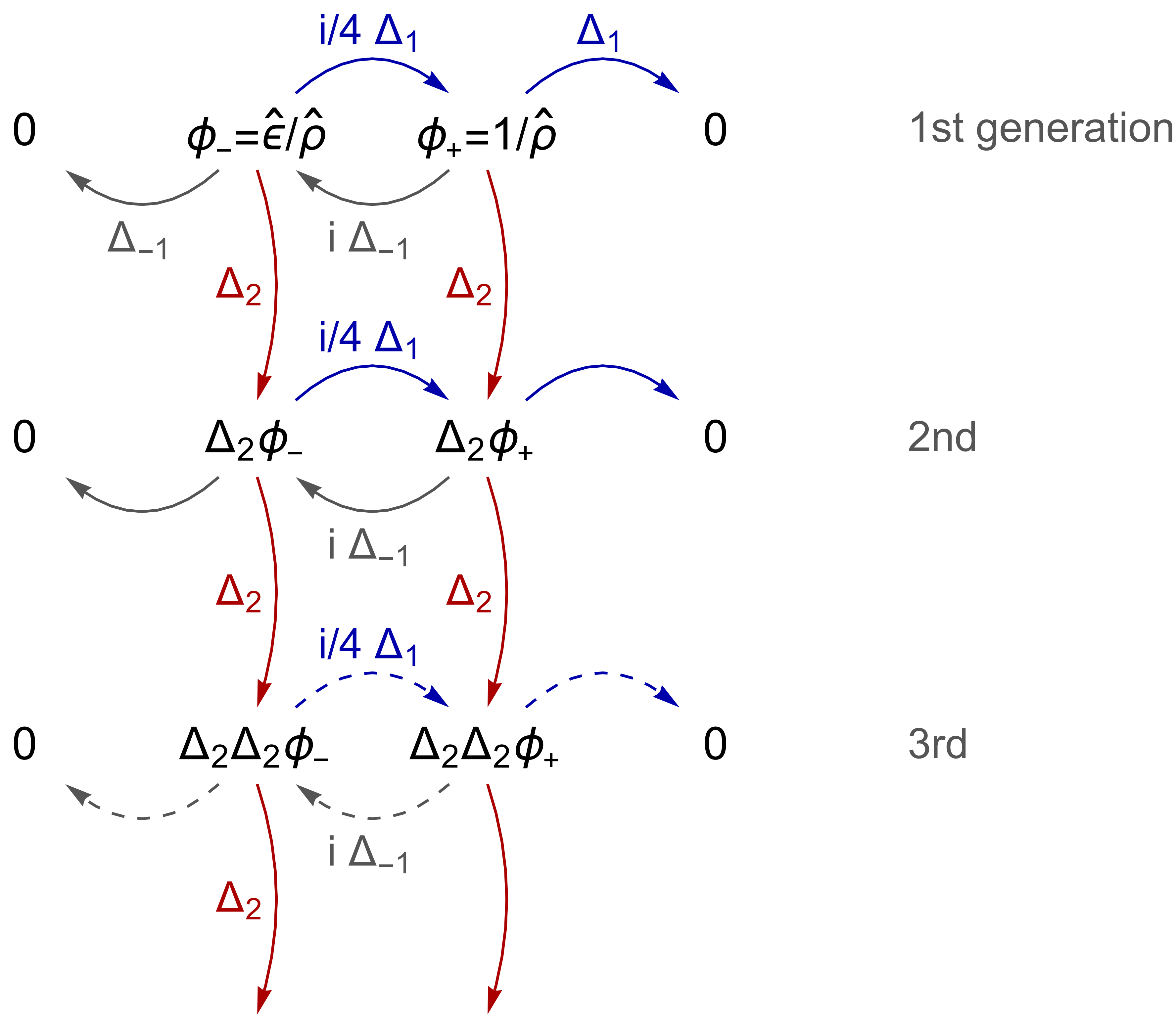
$\Delta_{-1}R = i(4\hat{\rho}E + \hat{\epsilon}R)$ $\quad \Delta_1 R = 0$ $\quad \Delta_2 R = -i/2\Big(1 + (\tfrac{1}{4}+a)/z + \ldots\Big)$

$\Delta_{-1}E = 2i\hat{\epsilon}E$ $\quad \Delta_1 E = 0$ $\quad \Delta_2 E = -i/2\Big(1 + 3/8z^2 + \ldots\Big)$



$E(B)$ $\quad e_1 = 1/4$ $\quad e_2 = 5/32 - a/2$ $\quad e_3 = \frac{57}{128} - \frac{5}{8}a + a^2$ $\quad$ **+ 60 terms**

Resurgence structure



Comparison with TBA

Median resummation

$$S_{\text{med}}(f) = S_+\Big(f + \frac{e^{-x}}{2}\Delta_1 f + \frac{e^{-2x}}{2}\Delta_2 f + \ldots + \frac{e^{-4x}}{8}\Delta_1\Delta_3 f + \frac{e^{-4x}}{8}\Delta_2^2 f + \ldots\Big)$$

$\Delta_2 R$

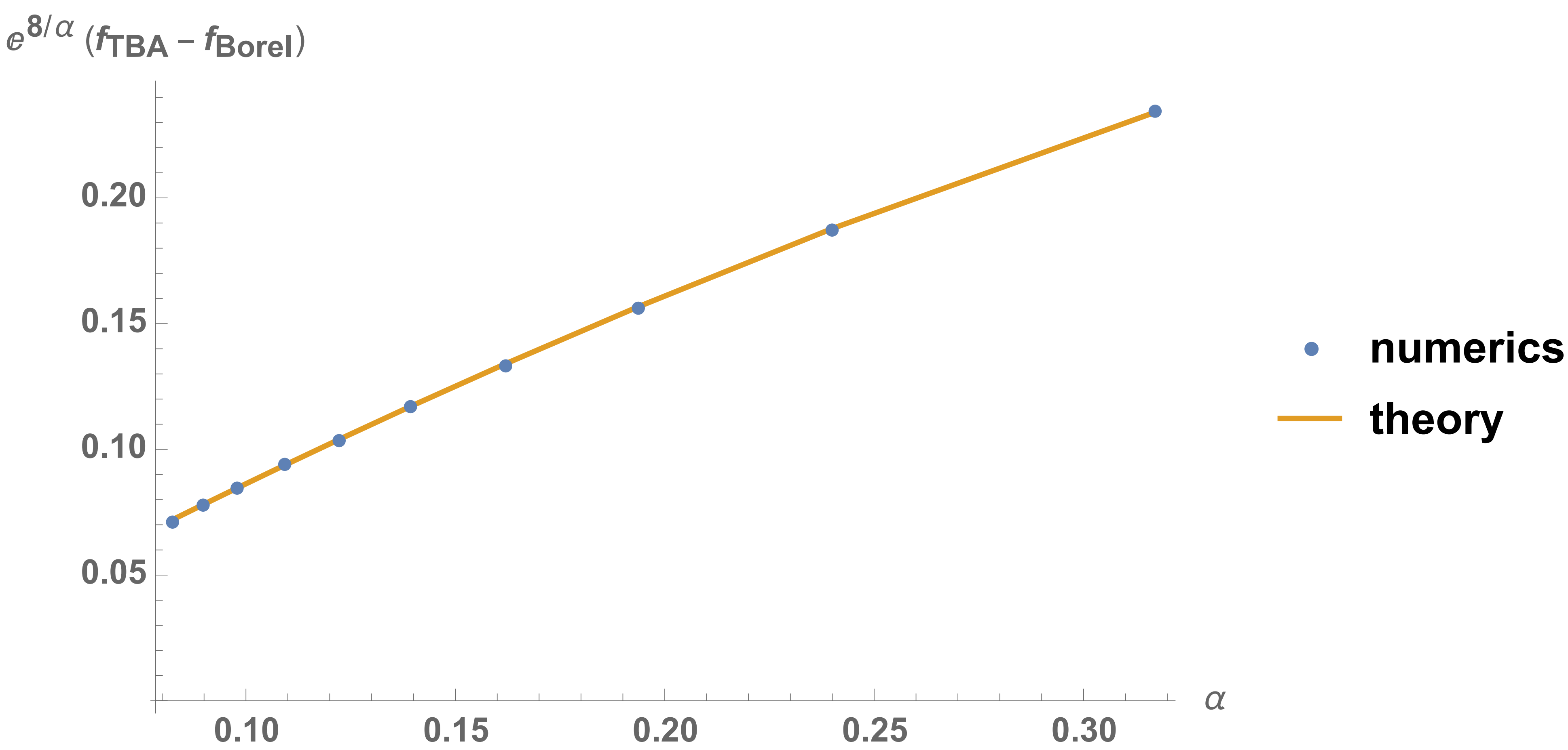
$\Delta_2 E$

$$f_{\text{Borel}} = S_{\text{med}}(f) = \mathfrak{Re}(S_+(f)) + \frac{32}{\mathrm{e}^4}\mathrm{e}^{-8/\alpha}(1 - \frac{5\alpha}{8} + \ldots)$$

$$d_1 = 0.58607 \qquad \frac{d_2}{d_1} = -0.6246$$

$$\frac{1}{\alpha} + \frac{1}{2} - B - \frac{1}{2}\log B\alpha = \log 2\hat{\rho}$$

We compare to the numerical solution of TBA



Trans-series

Analytic structure of the free energy on the Borel plane

The expansion of the physical observable is a trans-series

$$f_{\text{TBA}} = \sum_{m=0}^{\infty} e^{-\frac{2}{\alpha}m} \sum_{n=1}^{\infty} \chi_n^{(m)} \alpha^{n-1}$$

$\chi_n^{(0)} = \chi_n$

perturbative coefficients

$\mathrm{e}^{-2/\alpha}\chi_n^{(1)}$

$$\Delta_1 f = -\frac{16i}{\mathrm{e}^2}$$

$\mathrm{e}^{-4/\alpha}\chi_n^{(2)}$

$$\Delta_2 f = \frac{16i}{\mathrm{e}^2}\left(1 - \frac{3}{4x} + \frac{13}{32x^2} - \left(\frac{99}{256} - \frac{3}{8}\zeta_3\right)\frac{1}{x^3} + \ldots\right)$$

$\mathrm{e}^{-8/\alpha}\chi_n^{(4)}$

$$\Delta_2^2 f = \frac{32}{\mathrm{e}^4}\left(1 - \frac{5\alpha}{8} + \ldots\right)$$

large n asymptotics

large n asymptotics

Conclusions

- The integrable description enabled to calculate 2000 perturbative coefficient with high precision

- The asymptotic analysis of the perturbative coefficients revealed the analytic structure on the Borel plane with poles and cuts. The leading cuts showed a nice resurgence structure

- The various alien derivatives with the median resummation provided a trans-series ansatz, whose leading terms matched perfectly with the numerical solution of the TBA equation

- We recovered non-perturbative information from the perturbative series!

Some open problems

- Calculate analytically the numerically determined coefficients ✓
- Explain the resurgence structure
- Derive the trans-series ansatz
- Derive bridge equations which could connect the sectors
- Reveal the field theoretical origin of the terms in the trans-series
- Extend for other O(N) models (O(3) has instantons)
- Investigate the large N behaviour

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