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Bootstrap for Matrix Models and Lattice Yang-Mills Theory at large N

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Collaborations with Zechuan Zheng : arxiv:2108.04830
arxiv:2203.11360

Why Bootstrap?

Monte-Carlo is the only universal and systematic numerical method to compute functional integrals for (non-integrable) QFTs and Matrix Models.
This is (intellectually and practically) un unsatisfactory situation

Bootstrap opens the way for a new, more analytic approach

- Provides rigorous inequalities on physical quantities.
- Combines Schwinger-Dyson (loop) equations & positivity of correlation matrices
- Greatly inspired by conformal and S-matrix bootstrap
- Applicable to large variety of multiple and functional integrals.
- Large space for improvements
- Successfully applied to large N multi-matrix integrals
- Earlier attempts to bootstrap Lattice Yang-Mills theory

Rattazzi, Rychkov, Tonni
Paulos, Penedones, Toledo, van Rees, Vieira

Lin'20
Han, Hartnoll, Kruthoff'20
V.K., Zheng '21
De Mello Koch et al '21
Bhattacharya et al '21
Berenstein, Hulsay '22
Aikawa, Morita, Yoshimura'22

Jevicki, Karim, Rodriges, Levine 80's
Anderson, Kruszenski'17

Today:

1. We briefly review our application of bootstrap to a two matrix model V.K., Zheng '21
2. We report our results of bootstrap study of Lattice Yang-Mills theory in 2D, 3D and 4D and compare them to Monte-Carlo and perturbation theory V.K., Zheng '22

Bootstrap for Two-Matrix Model

H.Lin '20

V.K. & Zechuan Zheng '21

"Unsolvable" 2MM, with Hermitian $N \times N$ matrices $A, \; B$

$$Z = \lim_{N\to\infty} \int d^{N^2}A \, d^{N^2}B \, \mathrm{e}^{-N\mathrm{tr}V(A,B))}$$

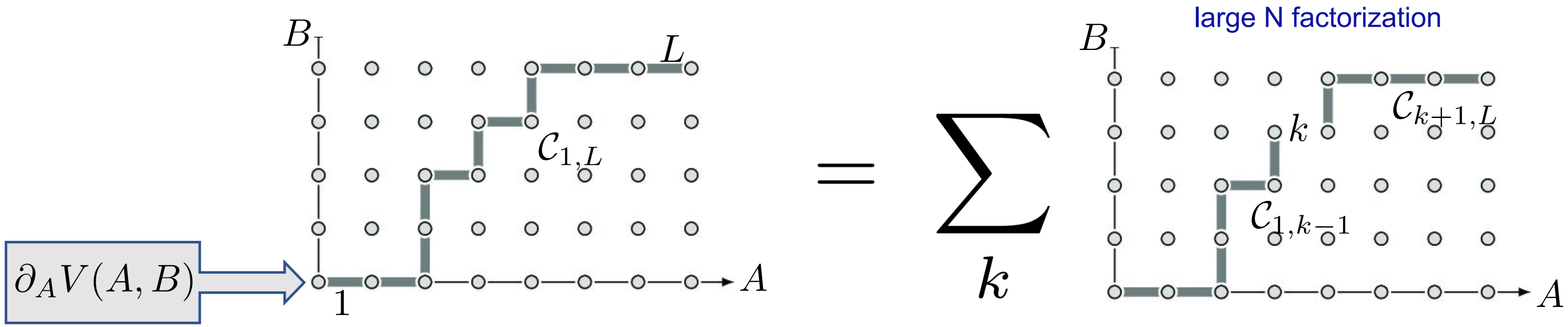
$$V(A,B) = -h[A,B]^2/2 + A^2/2 + gA^4/4 + B^2/2 + gB^4/4$$

Matrix "words": $\mathrm{Word} = ABBAAABAAB\ldots$

Schwinger-Dyson relations $\int d^{N^2}A \, d^{N^2}B \; \mathrm{tr}(\partial_A(\mathrm{Word} \times \mathrm{e}^{-N\mathrm{tr}\,V(A,B)}) = 0,$ (same for B)

give loop equations relating various moments : $\langle \mathrm{trWord} \rangle = \langle \mathrm{tr}(ABBAAABAAB\ldots) \rangle$

$$\langle \frac{\mathrm{tr}}{N} (\partial_A V(A,B) \; \times \mathrm{Word}(\mathcal{C}_{1L})) \rangle = \sum_{k=1}^{L} \delta_{k\sim A} \langle \frac{\mathrm{tr}}{N} \mathrm{Word}(\mathcal{C}_{1,k-1}) \rangle \cdot \langle \frac{\mathrm{tr}}{N} \mathrm{Word}(\mathcal{C}_{k+1,L}) \rangle$$



$$\partial_A V(A,B) \; = \; -2hBAB + hAB^2 + hB^2A + A + B + gA^3 + gB^3$$

For numerics, take (length of words) ≤ L_{max}. Many more words than equations!

How to complete the missing information? Positivity conditions on moments -- Matrix Bootstrap

Positivity of correlations and bootstrap

Linear combination of words of length ≤ L_{max} :

$$\mathcal{O} = \sum_{k=1}^{\mathcal{L}} \alpha_k \, \mathrm{Word}_k \; = \alpha_1 I + \alpha_2 A + \alpha_3 B + \alpha_4 A^2 + \alpha_5 B^2 + \alpha_7 AB + \alpha_8 BA + \alpha_9 ABB + \cdots$$

Positivity of the correlations:

$$\mathrm{Tr}\langle \mathcal{O}^\dagger \mathcal{O} \rangle = \sum_{k,j}^{\mathcal{L}} \alpha_k \, \alpha_j \, \mathbb{M}_{k,j} \; \geq 0 \,, \qquad \mathbb{M}_{k,j} = \langle \frac{1}{N} \mathrm{tr}(\mathrm{Word}_k \mathrm{Word}_j) \rangle \qquad \Longrightarrow \qquad \mathbb{M} \succeq 0$$

For the moments $\mathcal{W} = \{ I, \, \langle \mathrm{tr} A \rangle \, , \, \langle \mathrm{tr} B \rangle \, , \, \langle \mathrm{tr} A^2 \rangle , \, \langle \mathrm{tr} B^2 \rangle , \, \langle \mathrm{tr} AB \rangle \, , \, \langle \mathrm{tr} AB^2 \rangle , \, \langle \mathrm{tr} BA^2 \rangle , \, \langle \mathrm{tr} A^2 B^2 \rangle , \ldots \}$

the bootstrap process reduces to the following SDP (semi-definite programing):

$$\begin{aligned} &\text{minimize/maximize} \quad \sum_p c_p \, \mathcal{W}_p && \text{(computed observable)} \\ &\text{subject to} \quad \sum_p b_p^{(i)} \mathcal{W}_p = \sum_{pq} \mathcal{W}_p \mathcal{A}_{pq}^{(i)} \mathcal{W}_q && \text{(i'th loop equation),} \\ &\text{and} \quad \mathbb{M} \succeq 0 \,, && (\, \mathbb{M}_{ij} \; - \; \text{linear functions of } \mathcal{W}_p \,) \end{aligned}$$

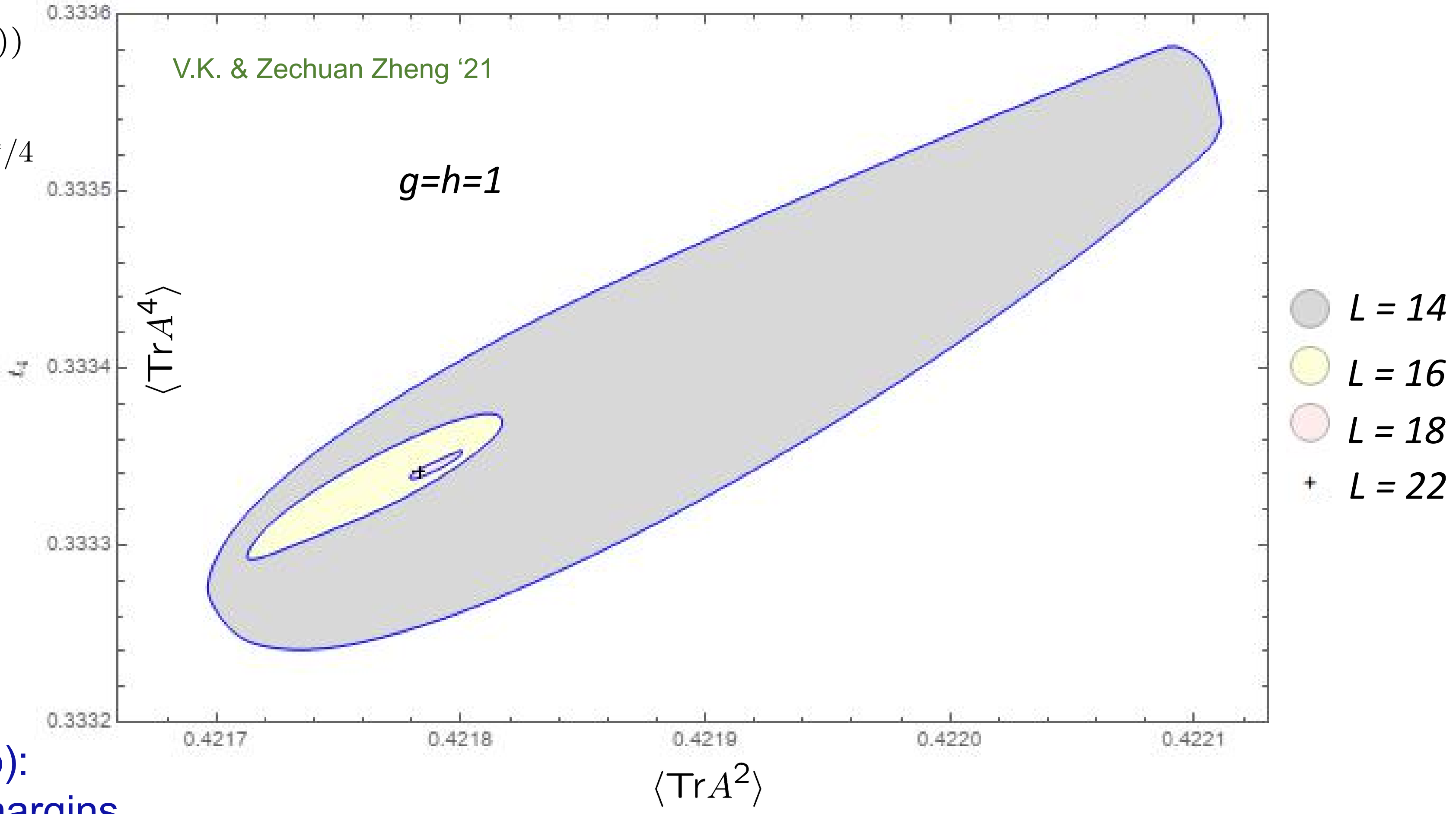
Loop eqs. are nonlinear -- the problem becomes highly non-convex.
To make it convex we apply the relaxation procedure (explained later)
It becomes a convex SDP (semi-definite programing), thus very efficient.

Numerical bootstrap for **2-matrix model**

$$Z = \lim_{N\to\infty} \int d^{N^2}A\, d^{N^2}B\, \mathrm{e}^{-N\mathrm{tr}V(A,B))}$$

$$V(A,B) = -h[A,B]^2/2 + A^2/2 + gA^4/4 + B^2/2 + gB^4/4$$

Functional integral!



Relaxed bootstrap gives 6-digit precision for cutoff $L_{max} = 22$

$$\begin{cases} 0.421783612 \le \langle \mathrm{Tr}A^2 \rangle \le 0.421784687 \\ 0.333341358 \le \langle \mathrm{Tr}A^4 \rangle \le 0.333342131 \end{cases}$$

Exact inequalities! (unlike Monte Carlo):
Increasing L_{max} will only improve the margins
Still within the capability of a single laptop: ~40 hours CPU time. Can be improved…

Compare to Monte Carlo (*N=800*): $\langle \frac{\mathrm{tr}}{N_c} A^2 \rangle = 0.42179(3)\,, \quad \langle \frac{\mathrm{tr}}{N_c} A^4 \rangle = 0.33336(5)$ (4 didgits, ~ 80 hours)

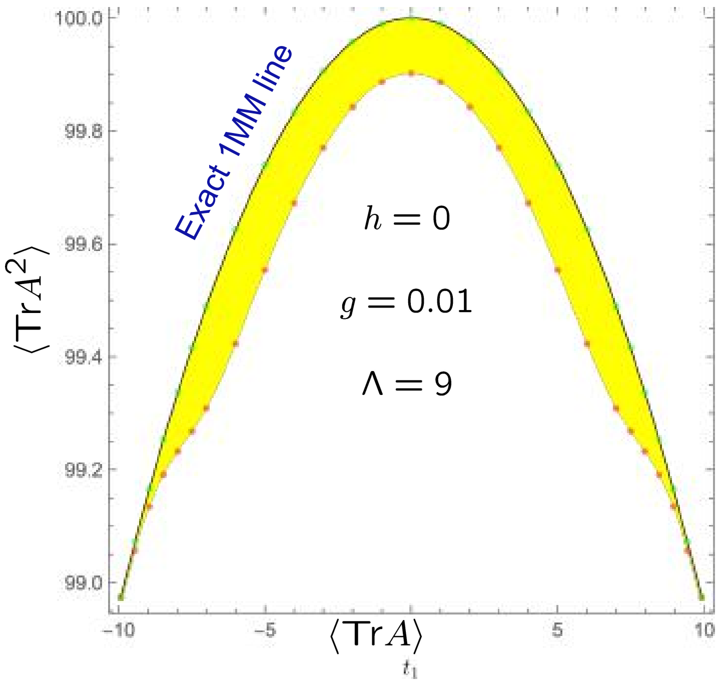
Roghav Govind Jha '21

Solutions with broken $\mathbb{Z}_2^{\otimes 3}$ symmetry $\langle \mathrm{Tr} A \rangle = \langle \mathrm{Tr} B \rangle \neq 0$

They exist only with negative signs of quadratic terms

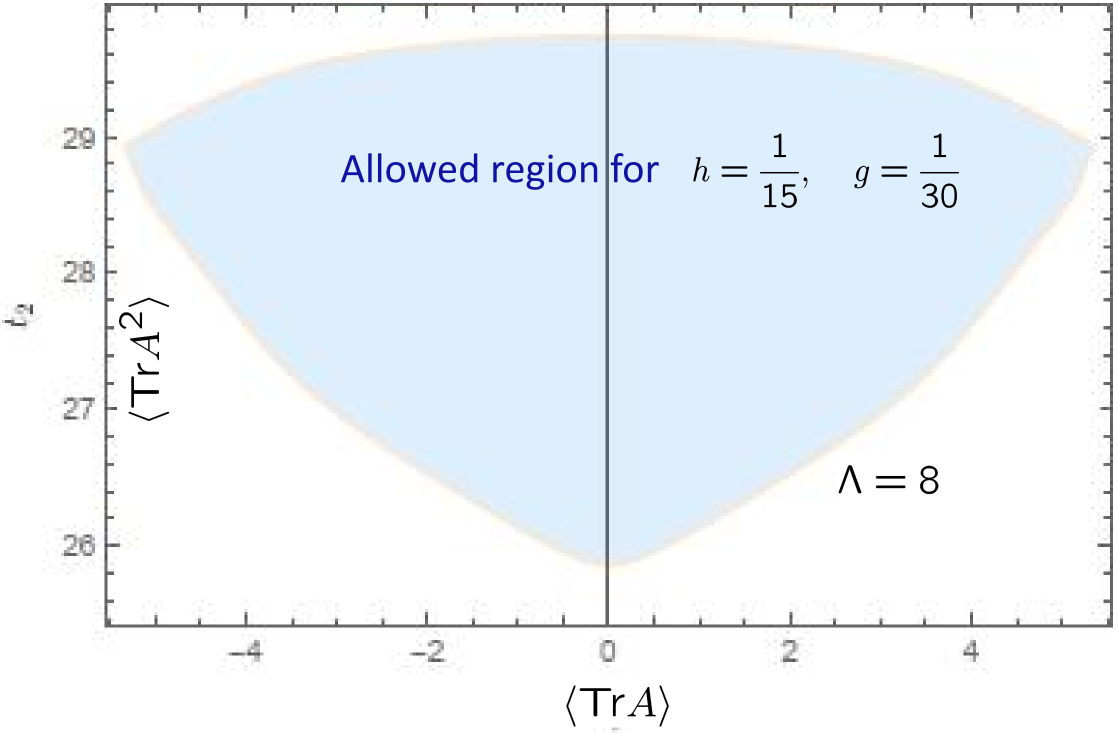
$$V(A,B) = \frac{N}{\hbar}\mathrm{tr}\left(-h[A,B]^2/2 - A^2/2 + gA^4/4 - B^2/2 + gB^4/4\right)$$

h=0: 2 decoupled 1MM's



Maximisation is much closer to the exact 1MM value

Numerical bootstrap for generic parameters



Upper limit seems to be the best estimate to exact line of symmetry breaking solutions

Bootstrap for Lattice Yang-Mills Theory ($N_c \rightarrow \infty$)

We solve Makeenko-Migdal loop equations for Wilson loops of lengths $\leq L_{max}$ supplemented by:

- Positivity and reflection-positivity of correlation matrices (unitarity of gauge variables)
- Relaxation of non-linear loop equations to convex inequalities
- Symmetry reductions of correlation matrices

We significantly modified the bootstrap scheme and improved the results (for plaquette average) w.r.t. the work of Anderson & Kruczenski '17.

Our numerical results look encouraging…

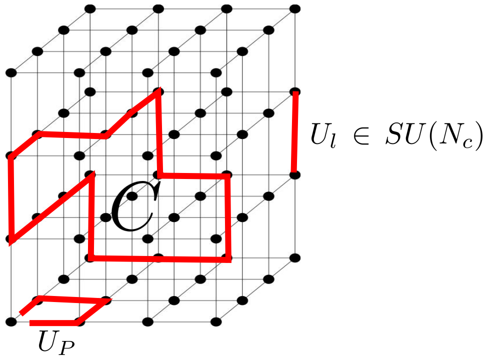
Makeenko-Migdal loop equation ($N_c \rightarrow \infty$)

Anderson, Kruczenski '17
V.K. & Zechuan Zheng '22

Lattice Yang-Mills, Wilson action $S = -\frac{N_c}{2\lambda} \sum_P \mathrm{tr} U_P$

Plaquette average $u_P = \frac{1}{N_c} \langle \mathrm{tr}\, U_P \rangle$

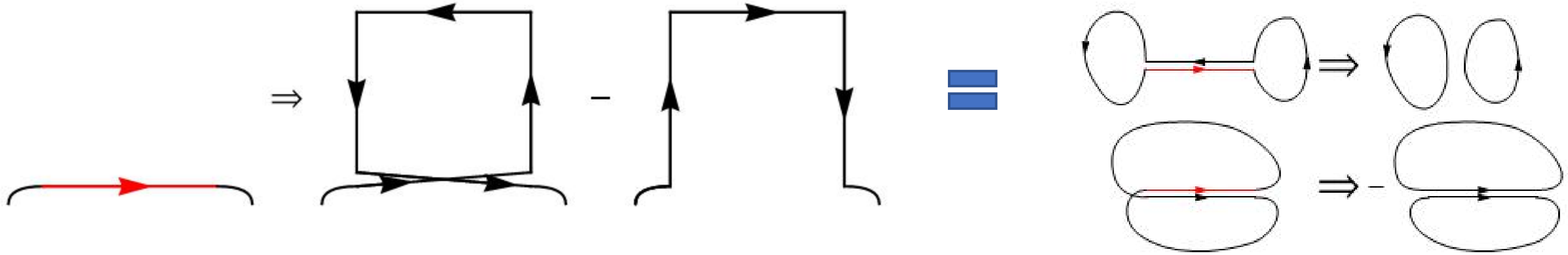
Wilson loop average $W[C] = \langle \frac{\mathrm{tr}}{N_c} \prod_{l \in C} U_l \rangle$



It satisfies Makeenko-Migdal loop equations (LE), which are Schwinger-Dyson equations obtained via the invariant shifts of every link variable: $U_l \rightarrow U_l (\mathbb{I} + i\epsilon)$

$\epsilon \in su(N_c)$

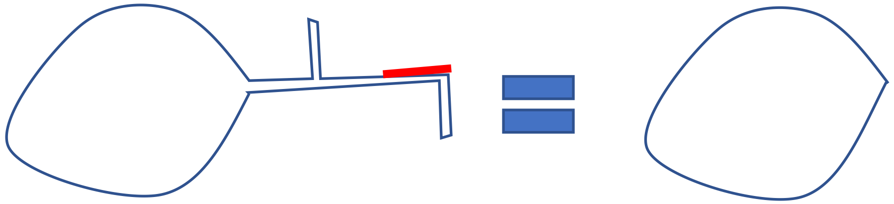
$$\sum_{\nu \perp \mu} \left(W[C_{l_\mu} \cdot \overrightarrow{\delta C^{\nu}_{l_\mu}}] - W[C_{l_\mu} \cdot \overleftarrow{\delta C^{\nu}_{l_\mu}}] \right) = \sum_{\substack{l' \in C \\ l' \sim l}} \epsilon_{l l'} W[C_{l l'}]\, W[C_{l' l}]$$



Back-track loop equations

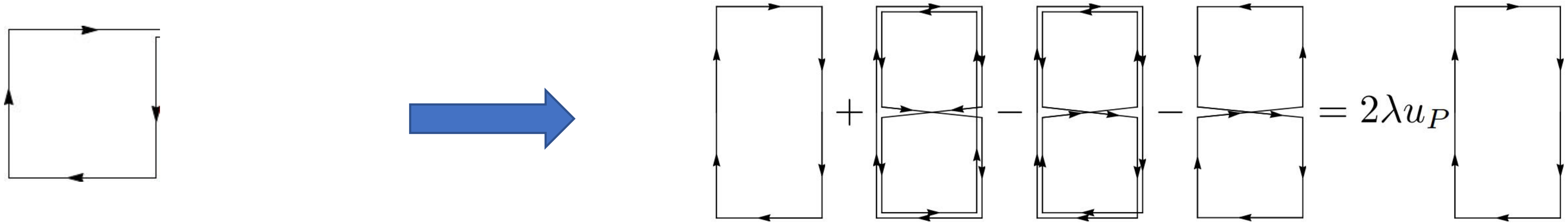
V.K. & Zechuan Zheng

Back-track identity on Wilson averages
(from unitarity of link variables)



One can write loop equations on back-track links.

Example of a non-linear back-track equation:



We use all loop equations on all Wilson averages for loops up to length ≤ L_{max}

$$\vec{W} = \{W[C_1], W[C_2], \ldots, W[C_{\mathcal{L}}]\}$$

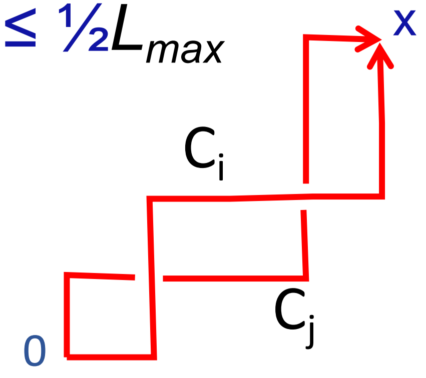
There are many more loops than equations. No clear boundary conditions for large loops.
We will impose instead a few positivity conditions

At L_{max}=16, for 3D and 4D, back-track LEs constitute more than 80% of all LEs. We have about 40,000 LEs in 3D and about 100,000 LEs in 4D. Around 3/4 of them are linear and independent. Only a minority are non-linear.

Positivity of correlation matrix

Jevicki, Karim, Rodriges, Levine 80's
Anderson, Kruszenski'17
V.K., Zheng '22

Define operator - linear combination of all Wilson paths (0→x) **of lengths ≤ ½L_{max}**

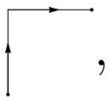
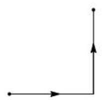


$$\mathcal{O} = \sum_{C_{0\to x}} \alpha[C_{0\to x}]\,\Psi[C_{0\to x}] \qquad \Psi[C_{0\to x}] = \prod_{l\in C_{0\to x}} U_l,$$

Positivity of quadratic form $\langle \mathrm{tr}\,(\mathcal{O}^\dagger \mathcal{O}\rangle \geq 0\,, \quad \forall\alpha$

Implies positive definiteness of correlation matrix $\mathbb{M}_{ij} = \langle \mathrm{tr}\,(\Psi(C_i)^\dagger \Psi(C_j))\rangle = \mathcal{W}(\bar{C}_i \circ C_j)$

$$\mathbb{M} \succeq 0$$

Example of correlation matrix, L_{max} =2+2, Paths 0 ⟶ (1,1): $\mathrm{Path}_1 =$  , $\mathrm{Path}_2 =$ 

$$\begin{array}{c} \\ \mathrm{Path}_1^\dagger \\ \mathrm{Path}_2^\dagger \end{array} \begin{array}{c} \begin{array}{cc} \mathrm{Path}_1 & \mathrm{Path}_2 \end{array} \\ \begin{pmatrix} 1 & u_P \\ u_P & 1 \end{pmatrix} \end{array} \succeq 0 \quad \Longrightarrow \quad u_P^2 \leq 1$$

unitarity constraint!

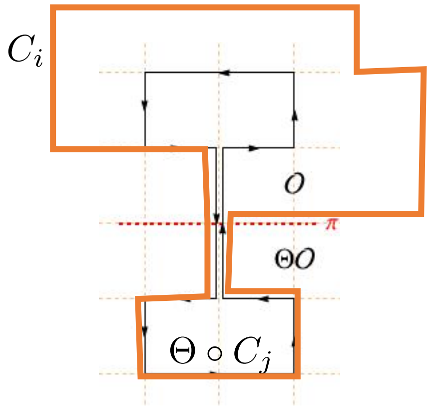
For our best data we take Paths 0 ⟶ 0.

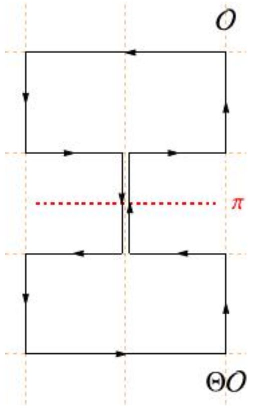
For L_{max} =∞ it would take into account all constraints. Works well even for L_{max} =16

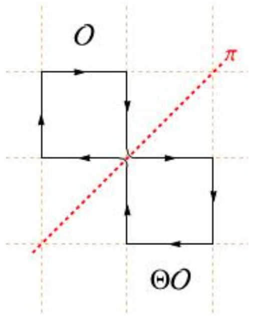
Reflection positivity of correlations

V.K. & Zechuan Zheng

Three types of reflection: site, link and diagonal, e.g.







Osterwalder, Seiler
Montvay, Munster

Positivity w.r.t. reflection Θ

$$\langle \mathrm{tr}\, [\mathcal{O}_+ \cdot (\Theta \circ \mathcal{O}_+)] \rangle \geq 0\,, \quad \forall \alpha$$

$$\mathbb{S} \succeq 0$$

$$\mathbb{S}_{ij} = \langle \mathrm{tr}\, (\Psi(C_i)\Psi(\Theta \circ C_j)) \rangle$$

Different from correlation matrix positivity and very important

Symmetry reduction

V.K. & Zechuan Zheng

- Lattice symmetries help to block-diagonalize the correlation and reflection matrices, greatly reducing the complexity of the problem. Symmetry for a matrix element:

$$\langle \mathrm{tr}\left(g \circ \mathcal{O}_1^\dagger\right) \cdot (g \circ \mathcal{O}_2)\rangle = \langle \mathrm{tr}\mathcal{O}_1^\dagger \cdot \mathcal{O}_2\rangle\,, \qquad g \in G' \subset \mathbb{Z}_2 \times \mathbb{Z} \times G_D$$

- Symmetries for Paths $0 \longrightarrow 0$ (B_D - hyper-octahedral group)

$$\mathcal{O} = \sum_{C_{0\to x}} \alpha[C_{0\to x}]\,\Psi[C_{0\to x}]$$

Dimension	Hermitian Conjugation	site&link reflection	diagonal reflection
2	$B_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
3	$B_3 \times \mathbb{Z}_2$	$B_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2^3$
4	$B_4 \times \mathbb{Z}_2$	$B_3 \times \mathbb{Z}_2$	$B_2 \times \mathbb{Z}_2^2$

Table 1: Invariant groups of correlation and reflection matrices

- If the vector space of paths decomposed into direct sum of irreps Rep_k of the invariant group with multiplicity m_k then positivity condition of the inner product matrix is equivalent to the collection of positivity conditions on the matrices corresponding to each Rep_k with matrix dimension $m_k \times m_k$

$$V = \bigoplus_{k=1}^{D} \mathrm{Rep}_k^{\bigoplus m_k}$$

- Example: for $D=3$, $L_{max}=16$, the $0 \longrightarrow 0$ correlation matrix is of huge size $(6505)^2$ Symmetry reduction to 20 smaller matrices of sizes 38, 15, 25, 18, 62, 33, 68, 75, 56, 78, 22, 18, 34, 15, 56, 33, 57, 76, 69, 73

Relaxation for matrix loop equations

V.K. & Zechuan Zheng '21

The Semi-Definite-Programing is highly non-convex for non-linear loop equations.
Idea: Replace the non-linear loop equations by linear relations + convex inequalities

We treat $X_{pq} = \mathcal{W}_p \, \mathcal{W}_q$ appearing in the r.h.s. of loop equations as independent variables

Relax it to condition of positive definiteness of the matrix

$$\mathbb{R} = \begin{pmatrix} 1 & \mathcal{W}_1 & \mathcal{W}_2 & \mathcal{W}_3 & \dots \\ \mathcal{W}_1 & X_{11} & X_{12} & X_{13} & \dots \\ \mathcal{W}_2 & X_{21} & X_{22} & X_{23} & \dots \\ \mathcal{W}_3 & X_{31} & X_{32} & X_{33} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \succeq 0$$

rank-1 matrix if $X_{pq} = \mathcal{W}_p \, \mathcal{W}_q$

We transformed the non-linear, non-convex problem to a convex one,
much more efficient for numerical SDP solvers!

The loss of information compensated by the increase of L_{max}
No clear understanding why it works so well…

Final SDP algorithm

V.K. & Zechuan Zheng

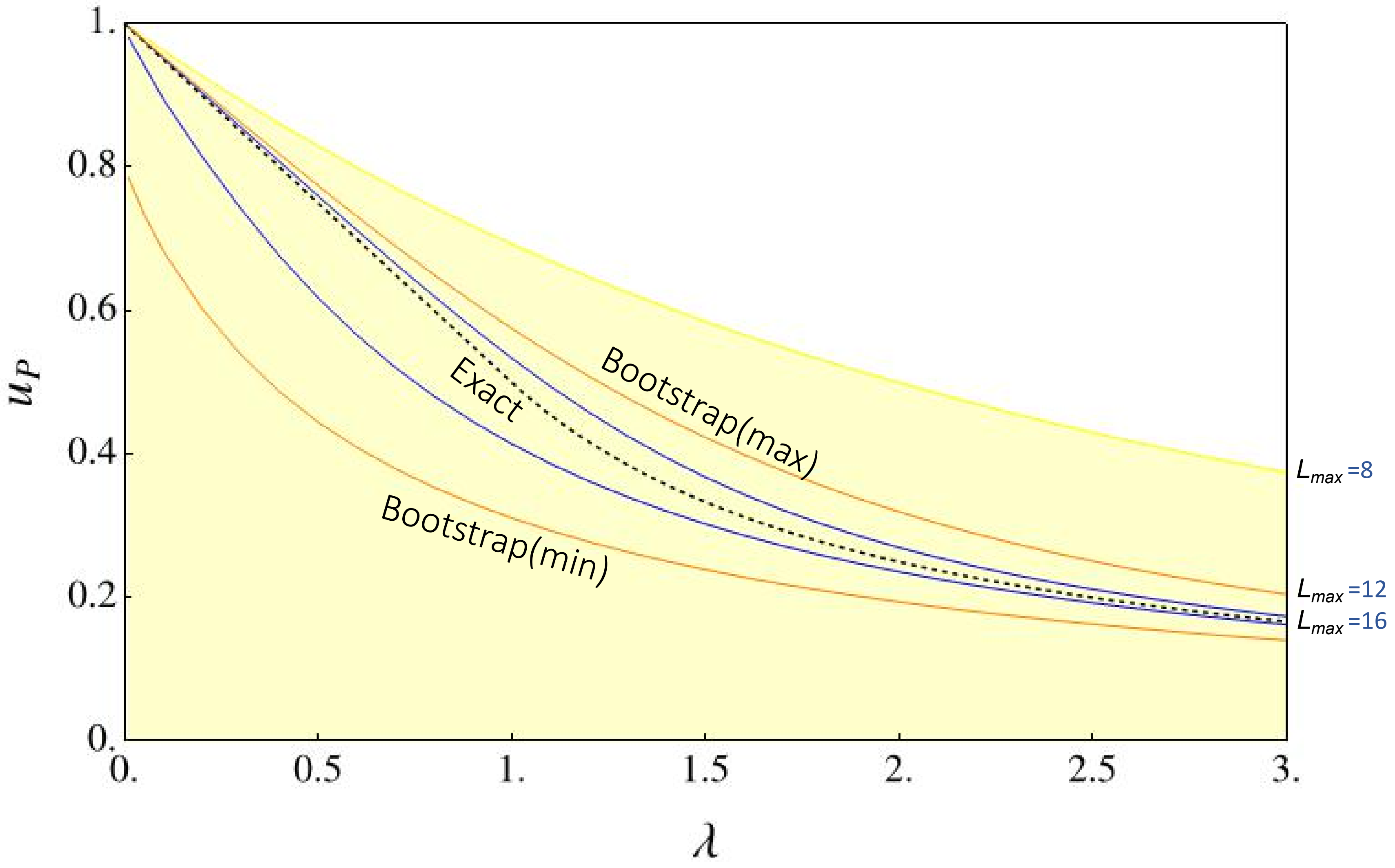
- Our bootstrap algorithm (similar to that for the 2-matrix model) :

$$
\begin{aligned}
&\text{minimize/maximize} && \sum_p c_p \, \mathcal{W}_p && \text{(computed observable)} \\
&\text{subject to} && \sum_p b_p^{(i)} \mathcal{W}_p = \sum_{pq} \mathcal{A}_{pq}^{(i)} \, X_{pq} && \text{(i'th loop equation)}, \\
&\text{and} && \mathbb{R} \succeq 0 \,, && \text{(relaxation condition $\mathcal{W}_p$)} \\
&\text{and} && \mathbb{M} \succeq 0 \,, && \text{(positivity of correlation matrix)} \\
&\text{and} && \mathbb{S}^{\text{site}} \succeq 0 \,, \quad \mathbb{S}^{\text{link}} \succeq 0 \,, \quad \mathbb{S}^{\text{diag}} \succeq 0 \,, && \text{(positivity of reflection matrices)} \\
& && \text{Symmetry block-reductions of all matrices} &&
\end{aligned}
$$

All conditions are linear or convex!

At the cutoff L_{max} =16, every data point takes ~20 hours of CPU time for 4D, and only half an hour for 3D (on a desktop computer)

Bootstrap for lattice Yang-Mills: <plaquette>(λ), $D=2$, $L_{max}=8, 12, 16$

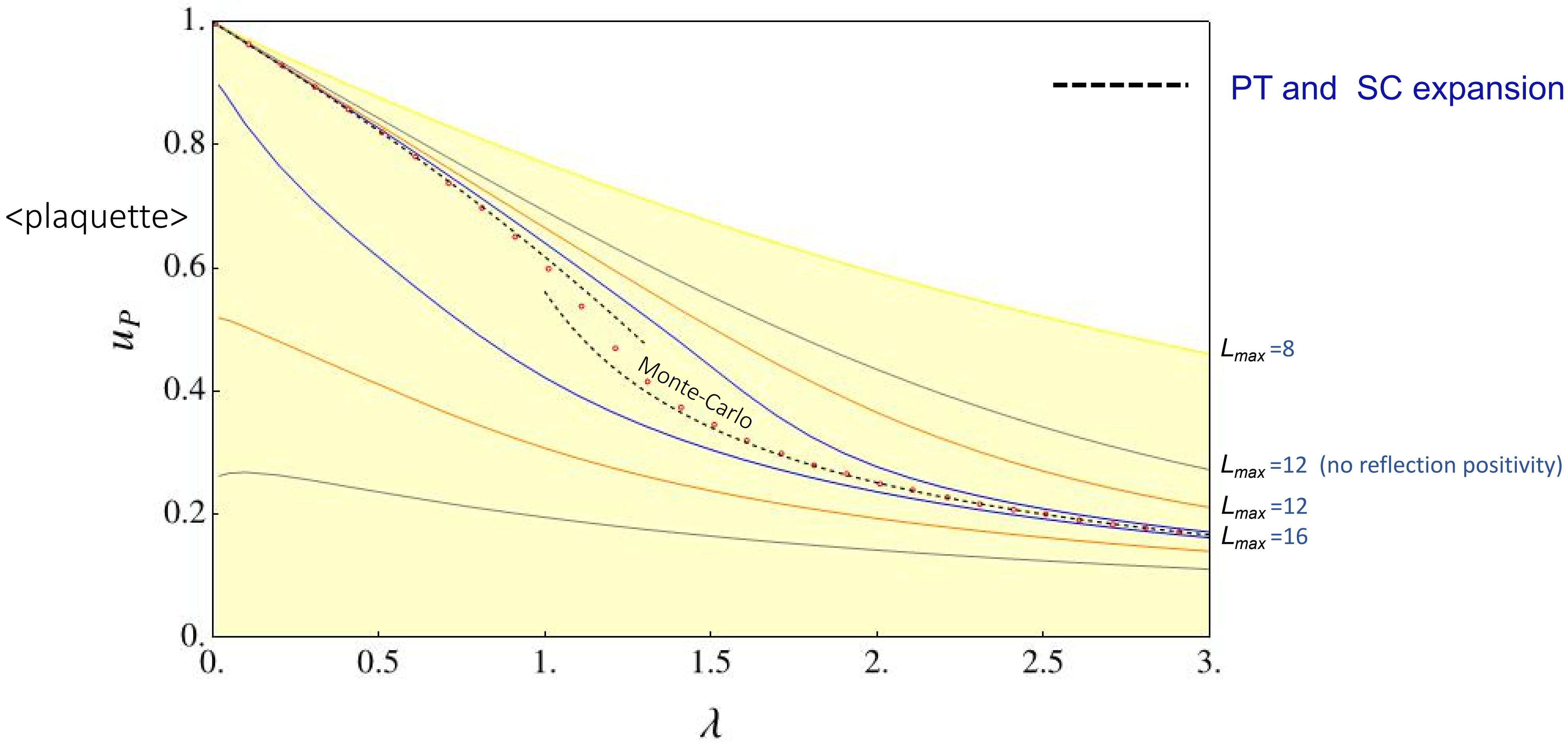


Exact solution:

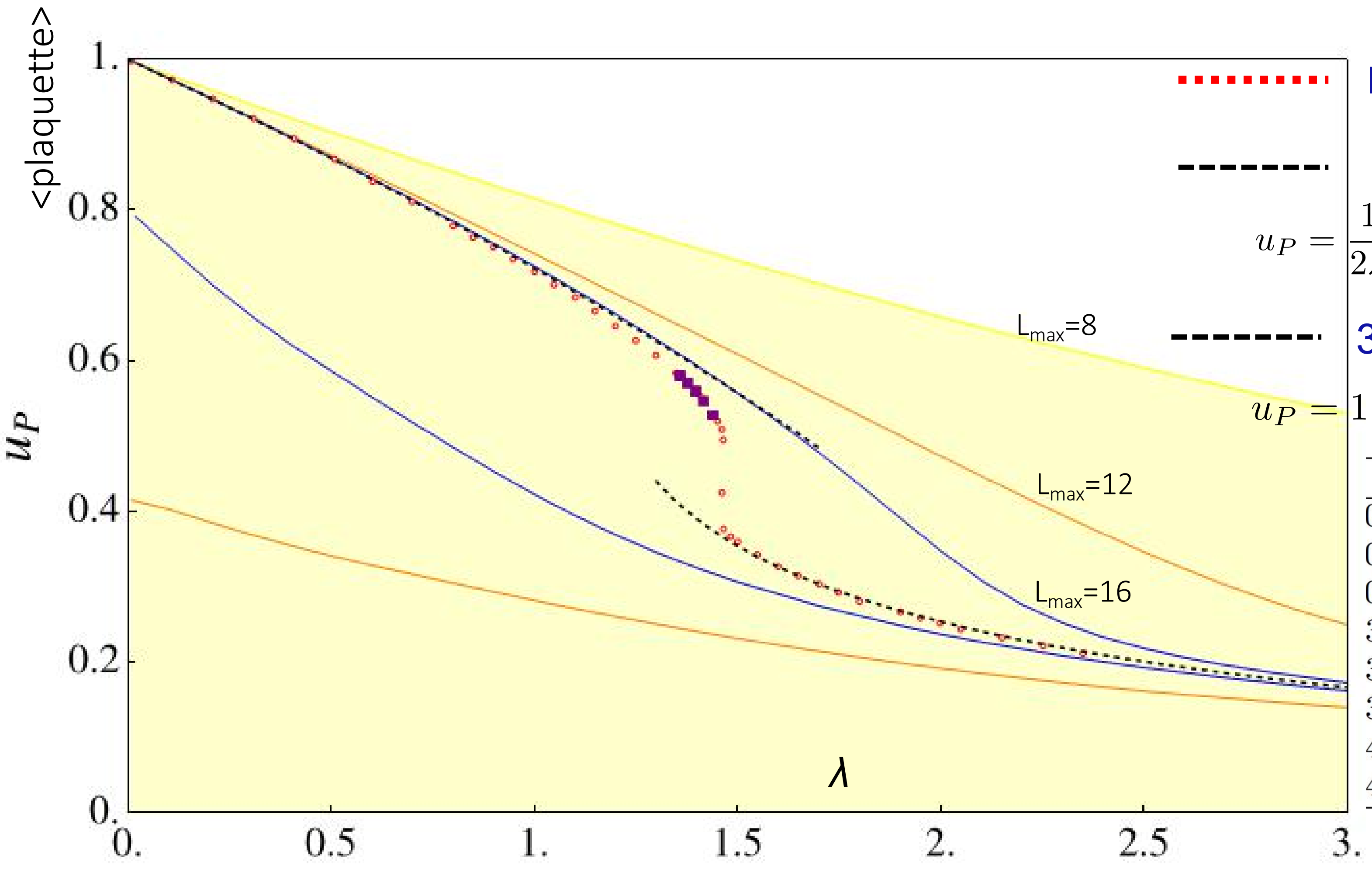
$$u_P = \begin{cases} 1-\frac{\lambda}{2}, & \text{for} \lambda \leq 1 \\ \frac{1}{2\lambda}, & \text{for} \lambda \geq 1 \end{cases}$$

Gross, Witten
Wadia

Bootstrap for Yang-Mills: <plaquette>(g), D=3, L_{max} =16



Main result: plaquette average $u_P = \frac{1}{N_c}\langle \mathrm{tr}\, U_P \rangle$ in D=4 lattice Yang-Mills



Monte-Carlo SU(10)

SU(10): Andersen, Kruczenski
SU(12): Athenodorou, Teper
SU(16),TEK: Gonzalez-Arroyo, Okawa

Strong coupling exp.: Drouffe, Zuber

$$u_P = \frac{1}{2\lambda} + \frac{1}{8\lambda^5} + \frac{15}{128\lambda^9} + \frac{17}{256\lambda^{11}} + \frac{273}{2048\lambda^{13}} + \mathcal{O}(\lambda^{-15})$$

3-loop perturbation theory: Alles Feo, Panagopoulos

$$u_P = 1 - \frac{\lambda}{4} - 0.020428\lambda^2 - 0.0063538\lambda^3 + \mathcal{O}(\lambda^8)$$

λ	$u_{p,\max}$	Perturbation	Monte Carlo
0.01	0.99750	0.997498	0.9975
0.31	0.92060	0.920348	0.9212
0.51	0.86706	0.866344	0.8676
35913	0.60837	0.606530	0.58043063
37733	0.60199	0.600313	0.57087665
39765	0.59480	0.593336	0.55936304
41802	0.58752	0.586302	0.5464461
44202	0.57886	0.577965	0.5275951

Bootstrap results: at IR cutoff L_{max}=16, upper bound close to 3-loop perturbation theory
lower bound close to strong coupling expansion

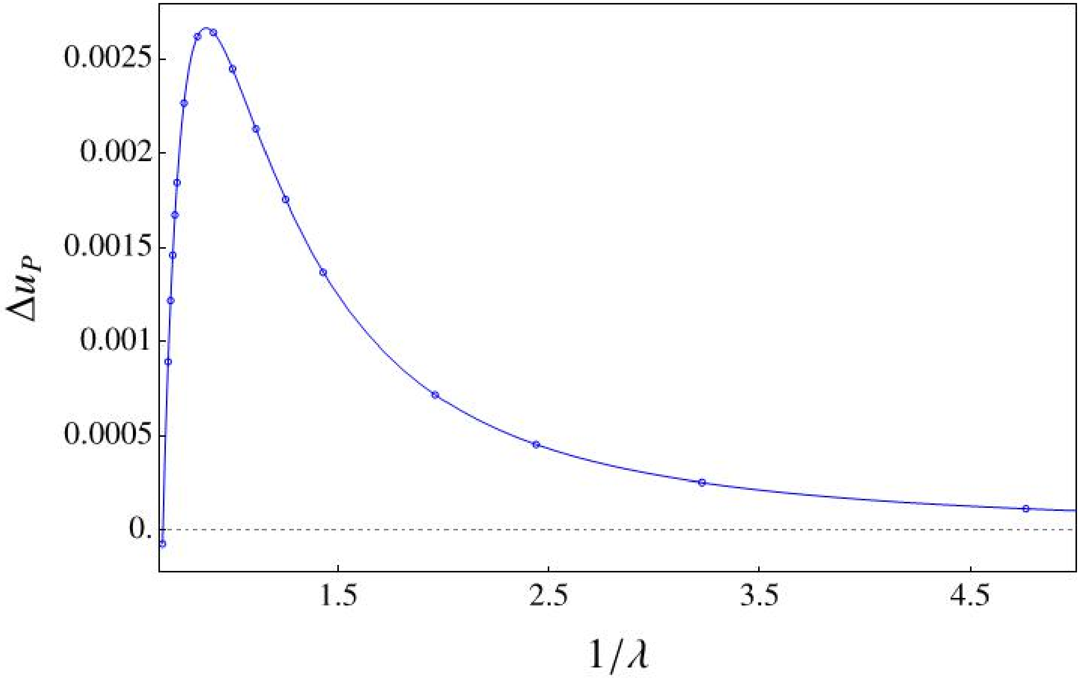
Not far from the Monte-Carlo data (if not too close to phase transition $\lambda \approx 1.5$)

Non-perturbative effects from <plaquette>(g) (D=4, L_{max}=16) ?

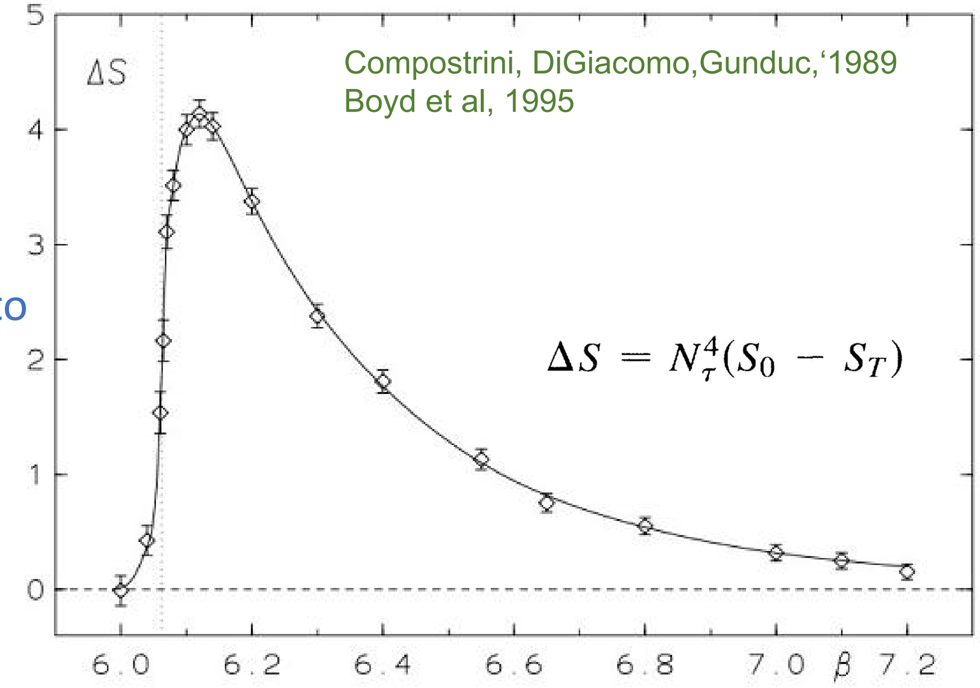
Attempt to measure non-perturbative effects for gluon condensate $\langle \mathrm{tr}\,(F_{\mu\nu}F^{\mu\nu})\rangle$

We take the difference of our plaquette average and 3-loop PT

$$\Delta u_P \equiv u_P^{\mathrm{boot}} - u_P^{\mathrm{PT}}$$



compare to



ΔS is the difference is vev's of the T=0 and T ≠ 0 plaquettes (MC N_c=3)

Our curve at T=0 is similar to the one of Boyd et al at finite T. Both originate from gluon condensate

Can be trusted only sufficiently far from 1st order phase transition point $\lambda \approx 1.5$
(but may be understood as analytic continuation ?)

Chernodub, V.K., Zheng (work in progress)

Basic open questions and future problems

- *Can we get a better precision than Monte-Carlo for D=3,4?*
- *Prospects of computing 1/N corrections? Linear problem!*
- *Finite N bootstrap?*
- *Quarks: We compute all Wilson loops! Sum them with spinorial factor*
- *Applications to other physical problems: Masses, S-matrix, real time processes, finite temperature barions, condensates…*
- *Flux-tube bootstrap?*
- *How to define gauge theory in terms of Wilson loops? What loop equations are independent?*

Thank You

My father (in the middle), advisor of the rector of Dalian Naval Institute (now Academy), with professors and his students, 1956