Two-dimensional massive integrable models on a torus

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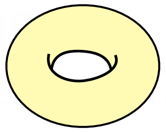
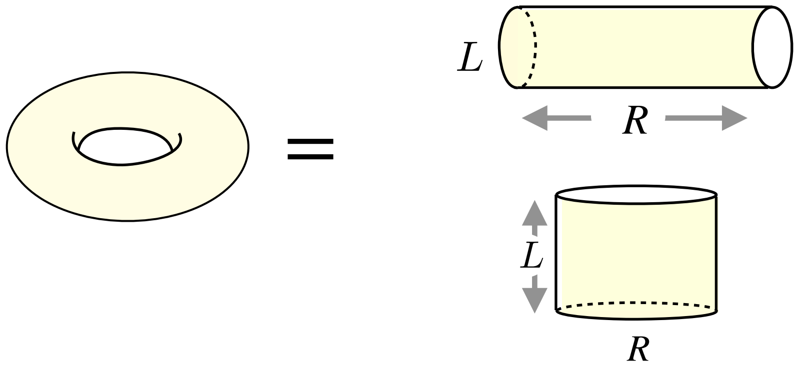


Shing-Tung Yau Center of Southeast University, September 13, 2022.

In this talk, I will address the following question:

How to compute the torus partition function (TPF) of an integrable 2D QFT knowing its S-matrix only?

- Hamiltonian approach: Assuming that the finite-volume spectrum is known, cut the torus along one of its cycles and insert a complete set of states



$$\mathcal{Z} = \sum_{\text{excitations } \in \mathcal{H}_L} e^{-R\text{Energy}}$$

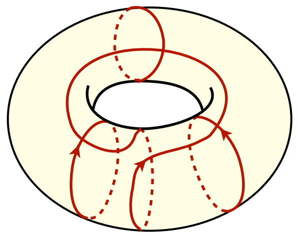
Hamiltonian evolution in the **direct** channel (along the R-cycle)

$$\mathcal{Z} = \sum_{\text{excitations } \in \mathcal{H}_R} e^{-L\text{Energy}}$$

Hamiltonian evolution in the **mirror** channel (along the L-cycle)

- Here I propose an alternative Loop gas approach: QFT as a grand canonical ensemble of interacting loops on the torus

$\mathcal{Z} =$

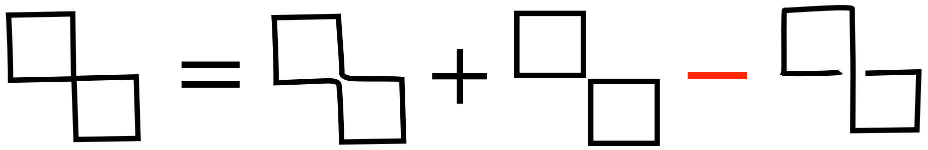


— measure = standard path integral for relativistic massive particle

— two-body interaction of loops produced by scattering factors associated with crossings

— The finite size effects are produced by the non-contractible loops characterised by a pair of winding numbers w and w'

- The loop gas approach generalises the representation [Vdovichenko} of the 2D Ising model as an ensemble of loops with minus signs associated with the crossings



Plan of the talk

I. Path integral for a loop on the torus

II. Loop-gas representation of free boson/fermion

III. Loop-gas representation of an interacting QFT on a torus.

1. Formulate the partition function as a gas of loops with two-body interactions associated with crossings

2. Decouple the interactions of loops by a Hubbard-Stratonovich transformation (essentially quantisation of the monodromies around the two cycles)

3. Perform the path integral over the loops to obtain an effective field theory for the HS fields.

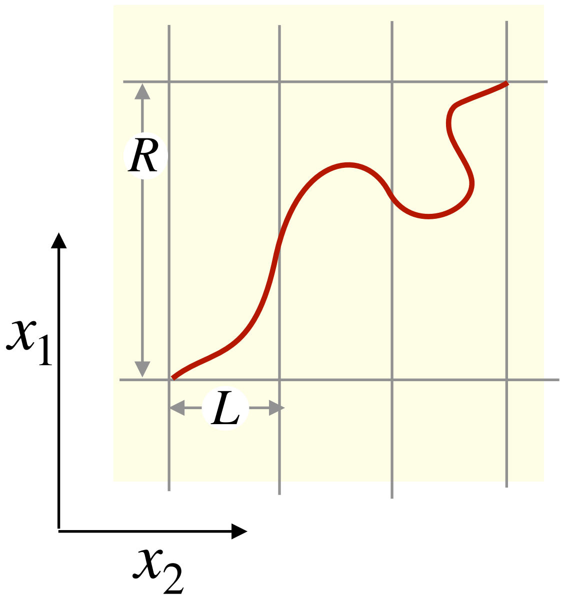
4. Oscillator representation

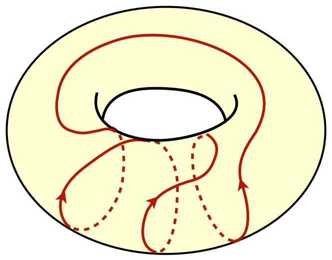
5. Mean field (classical) limits: $L \to \infty$ or $R \to \infty$. Relation to the Thermodynamical Bethe Ansatz.

IV. Example: The Sinh-Gordon Model

I. Path integral for loops

$\mathbb{T} = \mathbb{R}^2/\boldsymbol{\Omega}$ $\boldsymbol{\Omega} = L\mathbb{Z} \times R\mathbb{Z}$ - period lattice





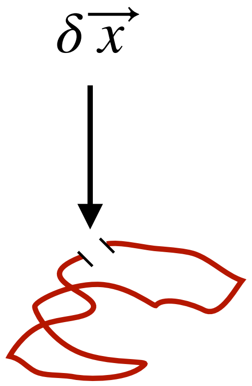
$w = 3,\ w' = 1$

The path integral for a loop on the torus includes a sum over the topological sectors characterised with winding numbers $w, w' \in \mathbb{Z}$

$$\mathscr{F} = \sum_{w,w' \in \mathbb{Z}} [\mathscr{F}]_{w,w'}$$

How to compute the path integral for a loop with winding numbers w, w' ?

Insert discontinuity $(\delta x_1, \delta x_2) = (w'R, wL)$ in the path integral for a closed loop:



$$[\mathscr{F}]_{w,w'} = \mathscr{F}(\overrightarrow{\delta x})\Big|_{\delta x_1 = w'R,\ \delta x_2 = wL}$$

$$\mathscr{F}(\overrightarrow{\delta x}) = -\tfrac{1}{2}\, RL \int \frac{d^2k}{(2\pi)^2} e^{i\vec{k}\cdot\overrightarrow{\delta x}} \log\left(\vec{k}^{\,2} + m^2\right)$$

For non-contractible loops, one of the integrals can be taken by residues:

$$\mathscr{F}(\Delta \overrightarrow{x}) = -\tfrac{1}{2}\, RL \int \frac{d^2k}{(2\pi)^2} \log\left(k_1^2 + k_2^2 + m^2\right)\, e^{ik_1\delta x_1 + ik_2\delta x_2}$$

$$= \frac{1}{2} \frac{R}{|\delta x_2|} \int_{\mathbb{R}} \frac{L dk_1}{2\pi} e^{ik_1\delta x_1 - \sqrt{k_1^2+m^2}|\delta x_2|} \qquad (\delta x_2 \neq 0)$$

wave function of on-shell particle in the **direct** channel analytically continued to imaginary time $t = -i\delta x_2$

$$= \frac{1}{2} \frac{L}{|\delta x_1|} \int_{\mathbb{R}} \frac{R dk_2}{2\pi} e^{ik_2\delta x_2 - \sqrt{k_2^2+m^2}|\delta x_1|} \qquad (\delta x_1 \neq 0)$$

wave function of on-shell particle in the **cross** channel analytically continued to imaginary time $t = -i\delta x_1$

The two integrals are related by a **mirror transformation** =
double Wick rotation exchanging the space and the time direction: $E \to -ip \qquad p \to iE$

or, in rapidity parametrization

$$p(\theta) = m\sinh(\theta)$$
$$E(\theta) = \sqrt{p^2+m^2} = m\cosh(\theta)$$

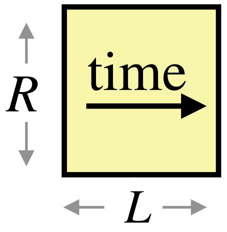
$$\theta \ \to \ i\pi/2 - \theta$$

Remarkably, the path integral over winding loops is expressed in terms of the wave functions of on-shell particles in the infinite spacetime. This will be used later to implement the scattering data.

Thus there are two possible descriptions of the winding loops:

- Description in **physical** kinematics (for loops winding at least once around the L-cycle):

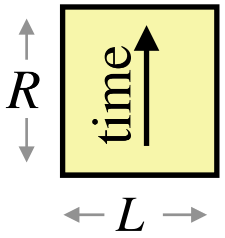
$$\mathscr{F}_{w,w'} = \frac{R}{2|w|}\int_{\mathbb{R}} \frac{dp(\theta)}{2\pi}\, e^{-|w|LE(\theta)+iw'Rp(\theta)} \qquad (w \neq 0)$$



on-mass-shell
winding particle in
physical kinematics

- Description in **mirror** kinematics (for loops winding at least once around the R-cycle):

$$\tilde{\mathscr{F}}_{w',w} = \frac{L}{2|w'|}\int_{\mathbb{R}} \frac{dp(\theta)}{2\pi}\, e^{-|w'|R\,E(\theta)+iwL\,p(\theta)} \qquad (w' \neq 0)$$



…
in the **mirror**
kinematics

$$\mathscr{F}_{w,w'} = \tilde{\mathscr{F}}_{w',w} \qquad (w, w' \neq 0)$$

Different choices for the kinematics lead to different but equivalent expressions for the free energy

IIa. Loop-gas representation for the free massive boson

No interaction - the sum exponentiates:

$$\mathcal{Z}_{\mathrm{FB}} = \exp[\mathcal{F}_{\mathrm{FB}}]$$

physical particles

mirror particles

one possible choice for the kinematics

$$\mathcal{F}_{\mathrm{FB}} = \sum_{w,w'\in\mathbb{Z}} [\mathcal{F}]_{w,w'} = \mathcal{F}_{0,0} + \sum_{w\neq 0} \mathcal{F}_{w,0} + \sum_{w'\neq 0, w\in\mathbb{Z}} \tilde{\mathcal{F}}_{w',w}$$

divergent infinite-volume energy density, to be neglected

$$\mathcal{F}_{\mathrm{FB}} = -\int_{\mathbb{R}} \frac{R\,dp(\theta)}{2\pi} \log\left(1 - e^{-LE(\theta)}\right) - \oint_{\mathscr{C}_{\mathbb{R}}} \frac{d\log\left(1 - e^{iLp(\theta)}\right)}{2\pi i} \log\left(1 - e^{-RE(\theta)}\right)$$

$$= \frac{\pi}{6}\frac{R}{L} c_0(mL) - \sum_{n\in\mathbb{Z}} \log(1 - e^{-RE_n})$$

$\mathscr{C}_{\mathbb{R}} = \{\overrightarrow{\mathbb{R} - i0}\} \cup \{\overleftarrow{\mathbb{R} + i0}\}$

= contour enclosing the real axis $\mathbb{R}$

effective central charge in the direct channel

$$c_0(mL) \equiv -\frac{6E}{\pi}\int_{\mathbb{R}} \frac{dp}{2\pi} \log\left(1 - e^{-L\sqrt{p^2+m^2}}\right)$$

excited states in the cross channel

$$p_n = \frac{2\pi n}{R} \qquad E_n = \sqrt{p_n^2 + m^2}$$

— Another choice of the kinematics leads to the dual formula with L and R exchanged

IIb. Loop-gas representation for the Ising Field Theory

$$\mathscr{Z}_{\text{IFT}} = \sum_{\text{loops}} (-1)^{\#\text{crossings}} \qquad (T > T_c)$$

$$(-1)^{\#\text{crossings}} = (-1)^{\#\text{loops}} \, (-1)^{w+w'} \, (-1)^{ww'} \qquad w = \sum w_i \, , \; w' = \sum w_i'$$

↑ two-body interaction of the loops

To disentangle the two-body interaction, introduce two discrete variables $\varepsilon, \varepsilon' \in \{0,1\}$:

$$(-1)^{w+w'+ww'} = \frac{1}{2} \sum_{\varepsilon,\varepsilon'=0,1} e^{\pi i(\varepsilon w + \varepsilon' w' + \varepsilon\varepsilon')}$$

$$\mathscr{Z}_{\text{IFT}} = \frac{1}{2} \sum_{\varepsilon,\varepsilon'} e^{i\pi\varepsilon\varepsilon'} D_{\varepsilon,\varepsilon'} = D_{1,1} + D_{0,1} + D_{1,0} - D_{0,0}$$

$$D_{\varepsilon,\varepsilon'} = \sum_{\text{loops}} e^{i\pi\varepsilon w + i\pi\varepsilon' w'} \, (-1)^{\#\text{loops}} \qquad (\varepsilon, \varepsilon' = 0,1)$$

The sum over loops in each of the terms exponentiates,

$$D_{\varepsilon,\varepsilon'} = \exp(\mathscr{F}_{\varepsilon,\varepsilon'}), \quad \mathscr{F}_{\varepsilon,\varepsilon'} = - \sum_{w,w' \in \mathbb{Z}} e^{i\pi(\varepsilon w + \varepsilon w')} \, [\mathscr{F}]_{w,w'}$$

$$\mathscr{F}_{\varepsilon,\varepsilon'} = R \int_{\mathbb{R}} \frac{dp(\theta)}{2\pi} \log\left(1 - e^{i\pi\varepsilon} e^{-LE(\theta}\right) + \oint_{\mathscr{C}_{\mathbb{R}}} \log\left(1 - e^{i\pi\varepsilon'} e^{-RE(\theta)}\right) \frac{d \log\left(1 - e^{i\pi\varepsilon} e^{iLp(\theta)}\right)}{2\pi i}$$

= partition function of massive Majorana fermion with boundary conditions ε and ε'

[Saleur-Itzykson 1987, Klassen-Melzer 1991]

Importantly, the partition function can be formulated solely in terms of the monodromies (phases) of the on-shell wave functions in the direct and in the cross channels through the functional

$$\mathcal{F} \overset{\text{def}}{=} -\int_{\mathbb{R}} \frac{d\phi(\theta)}{2\pi} \log\left(1 - e^{i\tilde{\phi}(i\pi/2-\theta)}\right) - \oint_{\mathscr{C}_{\mathbb{R}}} \frac{d\log\left(1 - e^{i\tilde{\phi}(\theta)}\right)}{2\pi i} \log\left(1 - e^{i\phi(i\pi/2-\theta)}\right)$$

$$= -\oint_{\mathbb{R}-i0} \frac{d\log\left(1 - e^{i\tilde{\phi}(\theta)}\right)}{2\pi i} \log\left(1 - e^{i\phi(i\pi/2-\theta)}\right) + \{L \leftrightarrow R\} \quad \text{(explicitly modular-invariant form)}$$

$$\mathcal{Z}_{\text{FB}} = \exp(\mathcal{F}) \qquad \text{with} \qquad \phi(\theta) = Rp(\theta),\ \ \tilde{\phi}(\theta) = Lp(\theta)$$

$$\mathcal{Z}_{\text{IFT}} = \frac{1}{2}\sum_{\varepsilon,\varepsilon'} e^{i\pi\varepsilon\varepsilon'} \exp(\mathcal{F}) \qquad \text{with} \qquad \phi(\theta) = Rp(\theta) + \pi\varepsilon,\ \ \tilde{\phi}(\theta) = Lp(\theta) + \pi\varepsilon'$$

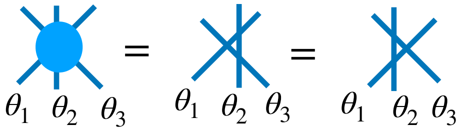
In the IFT, the sum over ε and ε' can be interpreted as expectation value $\mathcal{Z} = \langle \exp(\mathcal{F}) \rangle$ with
$\langle \phi(\theta) \rangle = Rp(\theta)$, $\langle \tilde{\phi}(\theta) \rangle = Lp(\theta)$, $\langle \phi\phi \rangle = \langle \tilde{\phi}\tilde{\phi} \rangle = 0$, $\langle \phi\tilde{\phi} \rangle = i\pi$.

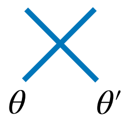
The claim is that in case of non-trivial scattering, the partition function can be cast in the same form, with the phases promoted to operators acting in an infinite-dimensional Hilbert space.

III. QFT’s with factorized scattering

(for simplicity one neutral particle, no bound states)

Factorized scattering:





$S(\theta-\theta')$ - two-particle scattering matrix

$S(\theta)S(-\theta)=1$	unitarity
$S(\theta)^*=S(-\theta^*)$	real analyticity
$S(\theta)=S(i\pi-\theta)$	crossing
$\sigma\equiv S(0)=\pm 1$	“TBA statistics”

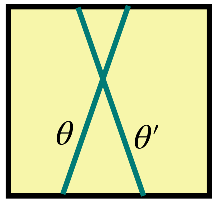
Mirror transformation exchanging the direct and the cross channels:

$$E(\theta)\to E(i\pi/2-\theta)=-ip(\theta)$$
$$p(\theta)\to p(i\pi/2-\theta)=iE(\theta)$$
$$\theta \;\to\; i\pi/2-\theta$$

III1.Loop gas formulation

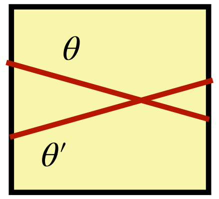
Claim: the torus partition function is given by the grand canonical ensemble of interacting loops, with scattering factors associated with the crossings.

Three types of crossings depending on the kinematics:



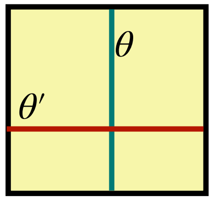
$S(\theta-\theta')$

physical/physical



$S(\theta-\theta')$

mirror/mirror



$W(\theta+\theta')$

physical/mirror

$$W(\theta)\equiv S(\theta-i\pi/2)$$

even real analytic function

$$W(\theta)=W(-\theta)=W(\theta+i\pi)^{-1}=W(\theta^*)^*$$

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III2. The phases for the monodromies as quantum fields

To disentangle the interactions due to scattering, upgrade the phases of the monodromies to gaussian fields whose two-point function produces the scattering phases + fermionic partners needed to obtain the correct measure over the rapidities (Gaudin determinant)

$$\phi(\theta) \;\rightarrow\; \Phi(\eta,\theta) = \phi(\theta) + \eta\psi(\theta)$$

$$\tilde{\phi}(\theta) \;\rightarrow\; \tilde{\Phi}(\tilde{\eta},\theta) = \tilde{\phi}(\theta) + \tilde{\eta}\tilde{\psi}(\theta)$$

$\eta, \tilde{\eta}$ - anticommuting variables

$\phi, \tilde{\phi}$ - boson gaussian fields

$\psi, \tilde{\psi}$- fermonic ghost fields

1- and 2-point functions:

$$\langle \Phi(\eta,\theta)\rangle = Lm\sinh\theta, \quad \langle \tilde{\Phi}(\tilde{\eta},\theta)\rangle = -Rm\sinh\theta$$

$$\langle \Phi(\eta,\theta)\tilde{\Phi}(\tilde{\eta},\theta')\rangle_c = -(1+\eta\tilde{\eta})\,\log W(\theta+\theta')$$

$$\langle \Phi\Phi\rangle_c = \langle \tilde{\Phi}\tilde{\Phi}\rangle_c = 0$$

Anti-periodic:

$$\Phi(\eta,\theta+i\pi) = -\Phi(\eta,\theta)$$

$$\tilde{\Phi}(\tilde{\eta},\theta+i\pi) = -\tilde{\Phi}(\tilde{\eta},\theta)$$

The integral over loops in the direct channel in the (w,w') sector

$$\mathbf{F}_{w,w'} = \frac{\sigma^{w+w'-1}}{2}\int d\eta d\tilde{\eta}\, e^{\eta\tilde{\eta}} \int_{\mathbb{R}} \frac{d\theta}{2\pi}\partial_\theta\Phi(\eta,\theta)\, \frac{\exp\left(i|w|\tilde{\Phi}(\tilde{\eta},i\pi/2-\theta)\right)}{|w|}\exp\left(iw'\Phi(\eta,\theta)\right)$$

$$= \int_{\mathbb{R}} \frac{d\theta}{2\pi}\; e^{i|w|\tilde{\phi}(i\pi/2-\theta)+iw'\phi(\theta)}\left(\frac{\partial_\theta\tilde{\phi}(\theta)}{|w|} + i\tilde{\psi}(i\pi/2-\theta)\partial_\theta\psi(\theta)\right) \qquad (w\neq 0)$$

and similarly in the cross channel $\tilde{\mathbf{F}}_{w,w'} = \ldots$

The sum over winding numbers exponentiates and the partition function takes the form

$$\mathscr{Z} = \sum_{N,\tilde{N}=0}^{\infty}\sum_{\{w_j\neq 0\}}\sum_{\{\tilde{w}_j\neq 0,\tilde{w}'_j\}} \int \frac{1}{N!\,\tilde{N}!}\left\langle \prod_{j=1}^{N}\mathbf{F}_{w_j,w'_j}\prod_{j=1}^{\tilde{N}}\tilde{\mathbf{F}}_{\tilde{w}'_j,\tilde{w}_j}\right\rangle = \left\langle \exp(\mathbf{F})\right\rangle$$

$$\mathbf{F} \overset{\text{def}}{=} -\int d\eta d\tilde{\eta}\, e^{\eta\tilde{\eta}}\int_{\mathbb{R}}\frac{d\Phi(\theta)}{2\pi}\;\log\left(1-\sigma e^{i\tilde{\Phi}(i\pi/2-\theta)}\right) - \oint_{\mathscr{C}_{\mathbb{R}}}\frac{d\log\left(1-\sigma e^{i\tilde{\Phi}(\theta)}\right)}{2\pi i}\log\left(1-\sigma e^{i\Phi(i\pi/2-\theta)}\right)$$

To summarise: The partition function on the torus is represented as the expectation value in an effective QFT for the phases ϕ and $\tilde{\phi}$ + fermonic partners

$$\mathscr{Z} = \left\langle \exp(\mathbf{F}) \right\rangle$$

$$\mathbf{F} \overset{\text{def}}{=} -\int d\eta d\tilde{\eta}\, e^{\eta\tilde{\eta}} \int_{\mathbb{R}} \frac{d\Phi(\theta)}{2\pi} \log\left(1-\sigma e^{i\tilde{\Phi}(i\pi/2-\theta)}\right) - \oint_{\mathscr{C}_{\mathbb{R}}} \frac{d\log\left(1-\sigma e^{i\tilde{\Phi}(\theta)}\right)}{2\pi i} \log\left(1-\sigma e^{i\Phi(i\pi/2-\theta)}\right)$$

Explicitly modular-invariant expression: $\mathbf{F} = -\int d\eta d\tilde{\eta}\, e^{\eta\tilde{\eta}} \oint_{\mathbb{R}-i0} \frac{d\log\left(1-e^{i\Phi(\theta)}\right)}{2\pi i} \log\left(1-\sigma e^{i\tilde{\Phi}(i\pi/2-\theta)}\right) + \{L \leftrightarrow R\}$

$$\Phi(\eta,\theta) = \phi(\theta) + \eta\psi(\theta)$$

$$\tilde{\Phi}(\tilde{\eta},\theta) = \tilde{\phi}(\theta) + \tilde{\eta}\tilde{\psi}(\theta)$$

Classical values:

$$\langle \Phi(\eta,\theta) \rangle = Lm\sinh\theta, \quad \langle \tilde{\Phi}(\tilde{\eta},\theta) \rangle = -Rm\sinh\theta$$

2-point functions:

$$\langle \Phi\Phi \rangle_c = \langle \tilde{\Phi}\tilde{\Phi} \rangle_c = 0$$

$$\langle \Phi(\eta,\theta)\tilde{\Phi}(\tilde{\eta},\theta') \rangle_c = -(1+\eta\tilde{\eta})\log W(\theta+\theta')$$

Symmetries:

$$\Phi(\eta,\theta+i\pi) = -\Phi(\eta,\theta)$$

$$\tilde{\Phi}(\tilde{\eta},\theta+i\pi) = -\tilde{\Phi}(\tilde{\eta},\theta)$$

$$W(\theta) = W(-\theta) = W(\theta+i\pi)^{-1} = W(\theta^*)^*$$

$$W(\theta) \equiv S(\theta - i\pi/2)$$

III4. Oscillator basis

$$\log W(\theta) = \sum_{k\geq 1,\mathrm{odd}}^{\infty} \frac{W_n}{n}\, e^{-n\theta} \qquad (\Re\theta > 0)$$

$$= \sum_{k\geq 1,\mathrm{odd}}^{\infty} \frac{W_n}{n}\, e^{n\theta} \qquad (\Re\theta < 0)$$

General form of the expansion for purely elastic scattering matrices [Klassen, Melzer, 1090]

Expansion in oscillators: $$\Phi(\eta,\theta) = \sum_{n\ \mathrm{odd}} i^{1-n}\Phi_n(\eta)\frac{e^{-n\theta}}{n}\,, \quad \tilde{\Phi}(\tilde{\eta},\theta) = \sum_{n\ \mathrm{odd}} i^{1+n}\,\tilde{\Phi}_n(\tilde{\eta})\frac{e^{n\theta}}{n}$$

$$[\Phi_n(\eta), \tilde{\Phi}_m(\tilde{\eta})] = -\,nW_n\delta_{m+n,0}\,(1+\eta\tilde{\eta})$$

The scattering matrix is encoded in the canonical commutation relations

$$\Phi_n(\eta) = \mathbf{a}_n + \eta\mathbf{b}_n\,, \quad \tilde{\Phi}_n(\tilde{\eta}) = \tilde{\mathbf{a}}_n + \tilde{\eta}\tilde{\mathbf{b}}_n)$$

bosonic fermionic

Fock vacuum: $\langle 0|0\rangle = 1$

$$\langle 0|\Phi_{-n} = \langle 0|\tilde{\Phi}_{-n} = 0\ \ ,\ \ \Phi_n|0\rangle = \tilde{\Phi}_n|0\rangle = 0 \quad (n>0,\ \ \mathrm{odd})$$

Fock-space representation of the torus partition function:

$$\mathscr{Z} = \langle 0|\,e^{\mathbf{H}_+}\,e^{\mathbf{F}}\,e^{-\mathbf{H}_-}|0\rangle$$

$$\mathbf{H}_- = \frac{m}{2W_1}\Big(L\tilde{\mathbf{a}}_{-1} + R\mathbf{a}_{-1}\Big)$$

$$\mathbf{H}_+ = \frac{m}{2W_1}\Big(L\tilde{\mathbf{a}}_{1} + R\mathbf{a}_{1}\Big)$$

The two periods of the torus are encoded in two “Hamiltonians” transforming the Fock vacua

III5. Mean-field limit $R \to \infty$ and relation to TBA

Take fermi TBA statistics, $\sigma = -1$

$$\mathcal{Z}_{\rm cyl} = \langle 0 | e^{\mathbf{H}_+} \, e^{\mathbf{F}_{\rm cyl}} \, e^{-\mathbf{H}_-} | 0 \rangle \qquad \mathbf{F}_{\rm cyl} \overset{\rm def}{=} - \int d\eta d\tilde{\eta} \, e^{\eta\tilde{\eta}} \int_{\mathbb{R}} \frac{d\Phi(\theta)}{2\pi} \log\left(1 + e^{i\tilde{\Phi}(i\pi/2-\theta)}\right)$$

Since the field $\phi(\theta)$ is Lagrange multiplier type, the field $\epsilon(\theta) \overset{\rm def}{=} i\tilde{\phi}(i\pi/2 - \theta)$ has no dispersion:

$$\left\langle \epsilon(\theta)\epsilon(\theta') \right\rangle_{\rm cyl} = \left\langle \epsilon(\theta) \right\rangle_{\rm cyl} \left\langle \epsilon(\theta') \right\rangle_{\rm cyl} \qquad \left\langle \mathcal{O} \right\rangle_{\rm cyl} \overset{\rm def}{=} \mathcal{Z}^{-1} \times \langle 0 | e^{\mathbf{H}_+} \, \mathcal{O} \, e^{\mathbf{F}_{\rm cyl}} \, e^{-\mathbf{H}_-} | 0 \rangle$$

Hence one can replace $\epsilon(\theta) \;\to\; \left\langle \epsilon(\theta) \right\rangle_{\rm cyl}$. No gaussian fluctuations, only tree Feynman graphs — pure mean field theory.

Dyson-Schwinger identities:

(1) $$\boxed{\epsilon(\theta) = LE(\theta) - \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} K(\theta - \theta') \; \log(1 + e^{-\epsilon(\theta')})}$$

(2) $$\boxed{\partial\phi(\theta) = R\partial p(\theta) + \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} K(\theta - \theta') \, \frac{\partial\phi(\theta')}{e^{\epsilon(\theta')} + 1}}$$

Relation to the TBA equations:

(1) ⟷ TBA equation for the pseudo energy

(2) ⟷ Bethe equations in terms of particle and hole densities $\rho_p(\theta) + \rho_h(\theta) = \partial\tilde{p}(\theta) + \int_{\mathbb{R}} \frac{d\theta'}{2\pi} K(\theta, \theta') \; \rho_p(\theta)$

upon the identification $\epsilon = \log \frac{\rho_h}{\rho_p}, \quad \partial_\theta \phi = R(\rho_p + \rho_h)$

The operator of the free energy is dispersion less as well: $\langle \exp(\mathbf{F}) \rangle_{\rm cyl} = \exp(\langle \mathbf{F} \rangle_{\rm cyl})$

$$\mathcal{Z}_{\rm cyl} = \exp[\mathcal{F}_{\rm cyl}], \quad \mathcal{F}_{\rm cyl} = \langle \mathbf{F}_{\rm cyl} \rangle_{\rm cyl} = R \int \frac{d\theta}{2\pi} \partial p(\theta) \; \log\left(1 + e^{-\epsilon(\theta)}\right)$$

IV. Example: Sinh-GORDON model

$$\mathscr{A} = \int_{\mathbb{T}} d^2x \left[\frac{1}{4\pi}(\nabla\phi)^2 + 2\mu\cosh(2b\phi)\right]$$

$$S(\theta) = \frac{\sinh(\theta) - i\sin(\pi\alpha)}{\sinh(\theta) + i\sin(\pi\alpha)}$$

$$\alpha = \frac{b^2}{1+b^2}$$

$$\log W(\theta) = \sum_{n\geq 1,\text{odd}} \frac{W_n}{n}\, e^{-n\theta}, \quad W_n = 4\cos\frac{n\pi a}{2}$$

$$a = 1 - 2\alpha = \frac{1-b^2}{1+b^2}$$

Remark 1: curiously the operator representation reproduces the infinite-volume energy density

$$\mathscr{Z}_{\text{tor}}^{(L,R)} \underset{R,L\to\infty}{=} \langle 0|\, e^{\mathbf{H}_+}\, e^{-\mathbf{H}_-}|0\rangle = \exp[LR\epsilon_0]$$

$$\epsilon_0 = \frac{m^2}{2W_1} = \frac{m^2}{8\sin\pi\alpha}\,.$$

[Destri-De Vega, 1991]

Remark 2: With this specific S-matrix one can write the Ward identity for φ as a finite-difference equation

$$\Big\langle \varphi(\theta + i\pi/2) + \varphi(\theta - i\pi/2)\Big\rangle_{\text{tor}} = \Big\langle \log(1 + e^{-\varphi(\theta + i\pi a/2)}) + \log(1 + e^{-\varphi(\theta - i\pi a/2)})\Big\rangle_{\text{tor}}$$

$$\big\langle \mathscr{O}\big\rangle_{\text{tor}} \overset{\text{def}}{=} \mathscr{Z}^{-1} \times \langle 0|\, e^{\mathbf{H}_+}\, \mathscr{O}\, e^{\mathbf{F}}\, e^{-\mathbf{H}_-}|0\rangle$$

Conclusion

The loop-gas method can be applied also for
— diagonal scattering matrices (type ADE)
— finite cylinder with integrable boundaries

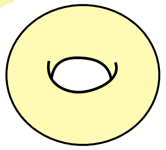
Generalisation to non-diagonal scattering and bound states not obvious - needs new insight

Not yet clear if this approach offers technical advantages, but one can try different things:

— systematic perturbative expansion above the mean field (TBA) limit

— Leclair-Mussardo formula for the torus (perturbatively in L/R)

— finite size effects in a QFT on a pair of pants

THANK YU!