Unitary matrix models, free fermion ensembles, and the giant graviton expansion

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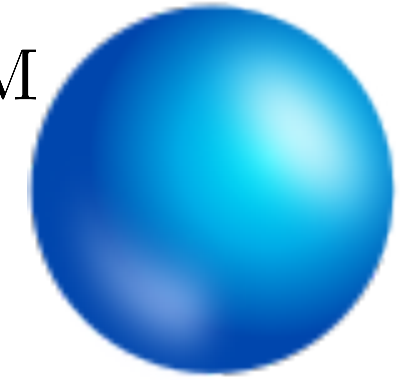
Mathematical set up and examples

Unitary matrix models (UMMs) capture the essence of gauge theory

- Free gauge theory/cmpct manifold

$\mathcal{N} = 4$ SYM

$U(N)$



S^3

Partition function

$\mathrm{Tr}_{\mathcal{H}_{SYM}}\, q^H$

Single-letter trace

$$f(q) = \mathrm{Tr}_{\mathcal{H}_{\text{single letters(SYM)}}}\, q^H$$

$$I_N^f(q) = \int dU \, \exp\left(\sum_{k=1}^{\infty} \frac{1}{k} f(q^k) \,\mathrm{tr}\, U^k \,\mathrm{tr}\, U^{-k} \right)$$

$$= \sum_{\ell=0}^{\infty} d_N^f(\ell)\, q^\ell$$

#(gauge invariant operators)

Supersymmetric indices of gauge theory are *equal* to UMMs (because they are protected)

- $\frac{1}{2}$-BPS $\quad \mathrm{Tr}\,(-1)^F\, q^{j_2}$

$$f_{1/2}(q) \;=\; q$$

- $\frac{1}{8}$-BPS $\quad \mathrm{Tr}\,(-1)^F\, q^{j_2 - j_1 + R}$

$$f_{1/8}(q) \;=\; \frac{2q}{1+q} \;=\; 2q - 2q^2 + 2q^3 - 2q^4 + 2q^5 - 2q^6 + \cdots$$

[Gadde, Rastelli, Razamat, Yan ’11; Bourdier, Drukker, Felix ’15]

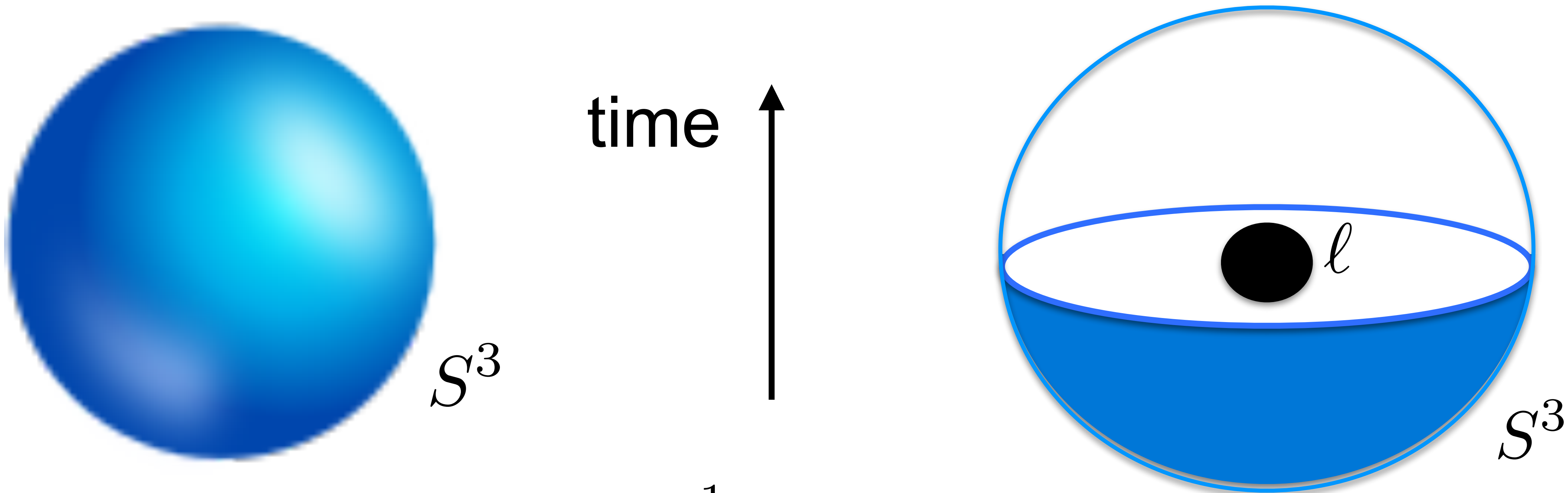
- $\frac{1}{16}$-BPS $\quad \mathrm{Tr}\,(-1)^F\, q^{j_2 + j_1 + r}$

$$f_{1/16}(q) \;=\; 1 - \frac{(1-q^2)^3}{(1-q^3)^2} \;=\; 3q^2 - 2q^3 - 3q^4 + 6q^5 - 2q^6 \;+\; \cdots$$

[Romelsberger ’05; Kinney, Maldacena, Minwalla, Raju ’05]

Physics context

The supersymmetric index has a dual interpretation according to AdS/CFT



$U(N)\ \mathcal{N}=4$ SYM $\qquad \frac{1}{N^2} = G \qquad \mathrm{AdS}_5\ (\times S^5)$

$\frac{1}{16}$-BPS states $\qquad\qquad \frac{1}{16}$-BPS BH

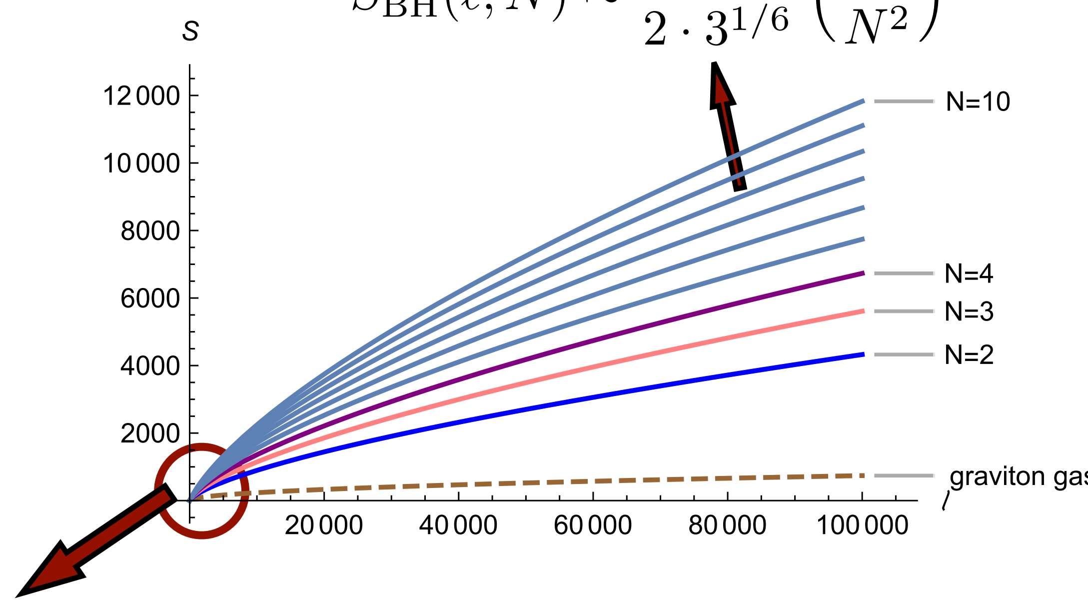
$\log d_N^{1/16}(\ell) \stackrel{?}{=} S_{\mathrm{BH}}(N,\ell)$	$S_{\mathrm{BH}}(N,\ell) = \dfrac{A_H(\ell)}{4G}$

[Sundborg '99; Aharony, Marsano, Minwalla, Papadodimas, van Raamsdonk '03; Kinney, Maldacena, Minwalla, Raju '05]

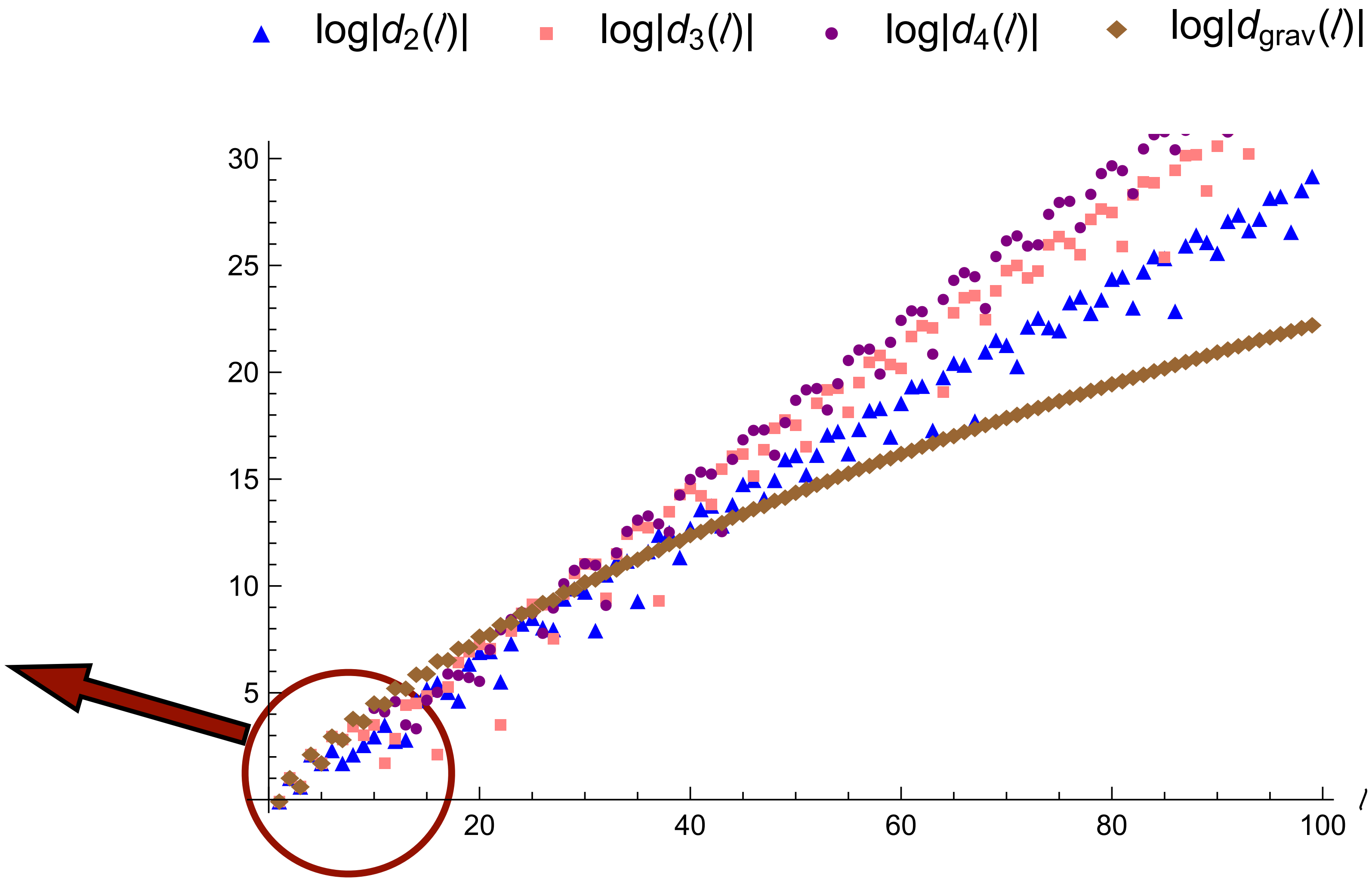
[Gutowski, Reall '04; Chong, Cvetic, Lu, Pope '05; Kunduri, Lucietti, Reall '06]

Large-charge asymptotics are captured by dual black hole

$$S_{\rm BH}(\ell, N) \sim \frac{\pi N^2}{2\cdot 3^{1/6}}\left(\frac{\ell}{N^2}\right)^{2/3}$$



$\frac{1}{16}$-BPS index



[Cabo-Bizet, Cassani, Martelli, S.M. ’18;
Choi, Kim, Kim, Nahmgoong ’18;
Benini, Milan ’18, ….]

[S.M. ’20]

What happens at small charges?

For fixed charge, the $\frac{1}{16}$-BPS index

$$I_N(q) \;=\; \sum_{\ell=0}^{\infty} d_N(\ell)\, q^{\ell}$$

[S.M. ’20]

$\frac{1}{16}$**-BPS index** $\quad f_{1/16}(q) \;=\; 1 - \frac{(1-q^2)^3}{(1-q^3)^2} \;=\; 3q^2 - 2q^3 - 3q^4 + 6q^5 - 2q^6 \;+\; \cdots$

ℓ	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$d_1(\ell)$	1	0	3	-2	3	0	0	6	-6	0	12	-18	27	-12	-27	60	-60

For fixed charge, the $\frac{1}{16}$-BPS index

[S.M. '20]

$$I_N(q) \;=\; \sum_{\ell=0}^{\infty} d_N(\ell)\, q^{\ell}$$

$\frac{1}{16}$**-BPS index** $\qquad f_{1/16}(q) \;=\; 1 - \dfrac{(1-q^2)^3}{(1-q^3)^2} \;=\; 3q^2 - 2q^3 - 3q^4 + 6q^5 - 2q^6 \;+\; \cdots$

ℓ	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$d_1(\ell)$	1	0	3	-2	3	0	0	6	-6	0	12	-18	27	-12	-27	60	-60
$d_2(\ell)$	1	0	3	-2	9	-6	11	-6	9	14	-21	36	-17	-18	114	-194	258

For fixed charge, the $\frac{1}{16}$-BPS index stabilizes as $N \to \infty$

[S.M. '20]

$$I_N(q) \;=\; \sum_{\ell=0}^{\infty} d_N(\ell)\, q^{\ell}$$

$\frac{1}{16}$**-BPS index** $\qquad f_{1/16}(q) \;=\; 1 - \dfrac{(1-q^2)^3}{(1-q^3)^2} \;=\; 3q^2 - 2q^3 - 3q^4 + 6q^5 - 2q^6 \;+\; \cdots$

ℓ	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$d_1(\ell)$	1	0	3	-2	3	0	0	6	-6	0	12	-18	27	-12	-27	60	-60
$d_2(\ell)$	1	0	3	-2	9	-6	11	-6	9	14	-21	36	-17	-18	114	-194	258
$d_3(\ell)$	1	0	3	-2	9	-6	21	-18	33	-22	36	6	-19	90	-99	138	-9
$d_4(\ell)$	1	0	3	-2	9	-6	21	-18	48	-42	78	-66	107	-36	30	114	-165
$d_5(\ell)$	1	0	3	-2	9	-6	21	-18	48	-42	99	-96	172	-156	252	-160	195
$d_6(\ell)$	1	0	3	-2	9	-6	21	-18	48	-42	99	-96	200	-198	345	-340	540
$d_7(\ell)$	1	0	3	-2	9	-6	21	-18	48	-42	99	-96	200	-198	381	-396	666
$d_\infty(\ell)$	1	0	3	-2	9	-6	21	-18	48	-42	99	-96	200	-198	381	-396	711

$= d_{\text{grav}}(\ell)$

This stability is a feature of all indices

$$I_N(q) \;=\; \sum_{\ell=0}^{\infty} d_N(\ell)\, q^\ell$$

$\frac{1}{8}$**-BPS index** $\qquad f_{1/8}(q) \;=\; \dfrac{2q}{1+q} \;=\; 2q - 2q^2 + 2q^3 - 2q^4 + 2q^5 - 2q^6 + \cdots$

ℓ	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$d_1(\ell)$	1	2	1	2	2	0	3	2	0	2	2	2	1	2	0	2	4
$d_2(\ell)$	1	2	4	4	6	8	8	8	13	12	12	16	14	16	24	16	18
$d_3(\ell)$	1	2	4	8	9	14	20	24	30	34	46	52	60	70	76	88	102
$d_4(\ell)$	1	2	4	8	14	18	28	40	52	70	88	104	140	168	196	240	278
$d_5(\ell)$	1	2	4	8	14	24	33	50	72	98	134	176	224	280	367	448	546
$d_6(\ell)$	1	2	4	8	14	24	40	56	84	122	168	232	312	408	528	672	865
$d_7(\ell)$	1	2	4	8	14	24	40	64	91	136	196	272	378	512	680	896	1162
$d_8(\ell)$	1	2	4	8	14	24	40	64	100	144	212	304	424	588	800	1072	1422
$d_9(\ell)$	1	2	4	8	14	24	40	64	100	154	221	322	460	640	886	1208	1622
$d_{10}(\ell)$	1	2	4	8	14	24	40	64	100	154	232	332	480	680	944	1304	1774
$d_{11}(\ell)$	1	2	4	8	14	24	40	64	100	154	232	344	491	702	988	1368	1880
$d_{\rm grav}(\ell)$	1	2	4	8	14	24	40	64	100	154	232	344	504	728	1040	1472	2062

This stability is a feature of all indices

$$I_N(q) \;=\; \sum_{\ell=0}^{\infty} d_N(\ell)\, q^{\ell}$$

$\frac{1}{2}$**-BPS index** $\qquad f_{1/2}(q) \;=\; q$

ℓ	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$d_1(\ell)$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$d_2(\ell)$	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9
$d_3(\ell)$	1	1	2	3	4	5	7	8	10	12	14	16	19	21	24	27	30
$d_4(\ell)$	1	1	2	3	5	6	9	11	15	18	23	27	34	39	47	54	64
$d_5(\ell)$	1	1	2	3	5	7	10	13	18	23	30	37	47	57	70	84	101
$d_6(\ell)$	1	1	2	3	5	7	11	14	20	26	35	44	58	71	90	110	136
$d_7(\ell)$	1	1	2	3	5	7	11	15	21	28	38	49	65	82	105	131	164
$d_8(\ell)$	1	1	2	3	5	7	11	15	22	29	40	52	70	89	116	146	186
$d_9(\ell)$	1	1	2	3	5	7	11	15	22	30	41	54	73	94	123	157	201
$d_{10}(\ell)$	1	1	2	3	5	7	11	15	22	30	42	55	75	97	128	164	212
$d_{11}(\ell)$	1	1	2	3	5	7	11	15	22	30	42	56	76	99	131	169	219
$d_{\rm grav}(\ell)$	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	231

In fact, this stability is a feature of all UMMs

- $$\mathcal{I}_N^f(q) = \int dU \, \exp\left(\sum_{k=1}^{\infty} \frac{1}{k} f(q^k) \, \mathrm{tr}\, U^k \, \mathrm{tr}\, U^{-k} \right)$$

Character expansion,
Frobenius formula

$$\mathcal{I}_N^f(q) = \sum_{\lambda} f_\lambda(q) \frac{1}{z_\lambda} \sum_{\ell(\mu) \leq N} \chi^\mu(\lambda)^2$$

Partitions of integers

$$\lambda = 1^{r_1} \, 2^{r_2} \, \ldots$$

$$z_\lambda = \prod_{i \geq 1} r_i! \, i^{r_i}$$

$$f_\lambda(q) = \prod_{i \geq 1} f(q^i)^{r_i}$$

=1

- At $N = \infty$, we sum over all partitions, and

$$\Rightarrow \quad \mathcal{I}_\infty^f(q) = \sum_{\lambda} f_\lambda(q) = \prod_{k=1}^{\infty} \frac{1}{1 - f(q^k)} = \mathcal{I}_{\mathrm{grav}}^f(q)$$

[S.M. '20] [cf Dolan '07; Dutta, Gopakumar '07]

What happens above the stability bound?

$\frac{1}{2}$-BPS index $\qquad \dfrac{I_N(q)}{I_\infty(q)} = 1 + q^{N+1} \sum_{\ell=0}^{\infty} d_N^{(1)}(\ell)\, q^\ell$

[Imamura + Arai, Fujiwara, Mori, '19-'21]
[Gaiotto-Lee '21; Lee '22]

ℓ	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$d_1^{(1)}(\ell)$	-1	-1	-1	0	0	1	1	1	1	1	0	0	0	-1	-1	-1	-1
$d_2^{(1)}(\ell)$	-1	-1	-1	-1	0	0	1	1	2	1	2	1	1	0	0	-1	-1
$d_3^{(1)}(\ell)$	-1	-1	-1	-1	-1	0	0	1	1	2	2	2	2	2	1	1	0
$d_4^{(1)}(\ell)$	-1	-1	-1	-1	-1	-1	0	0	1	1	2	2	3	2	3	2	2
$d_5^{(1)}(\ell)$	-1	-1	-1	-1	-1	-1	-1	0	0	1	1	2	2	3	3	3	3
$d_6^{(1)}(\ell)$	-1	-1	-1	-1	-1	-1	-1	-1	0	0	1	1	2	2	3	3	4
$d_7^{(1)}(\ell)$	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	0	1	1	2	2	3	3
$d_8^{(1)}(\ell)$	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	0	1	1	2	2	3
$d_9^{(1)}(\ell)$	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	0	1	1	2	2
$d_{10}^{(1)}(\ell)$	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	0	1	1	2
$d_{11}^{(1)}(\ell)$	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	0	1	1
$d_{12}^{(1)}(\ell)$	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	0	1

Example 1: $\frac{1}{2}$-BPS index

$$f_{1/2}(q) \;=\; q$$

$$I_N^{1/2}(q) \;=\; \frac{1}{(q)_N}$$

$$(q)_N \;=\; \prod_{i=1}^{N}(1-q^i)$$

$$\frac{I_N^{1/2}(q)}{I_\infty^{1/2}(q)} \;=\; \frac{(q)_\infty}{(q)_N}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m \, q^{\binom{m}{2}}}{(q)_m} \, q^{m(N+1)}$$

The higher coefficients do not always stabilize to constant numbers

$\frac{1}{8}$**-BPS index**

$$\frac{I_N(q)}{I_\infty(q)} = 1 + q^{N+1} \sum_{\ell=0}^{\infty} d_N^{(1)}(\ell)\, q^\ell$$

ℓ	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$d_1^{(1)}(\ell)$	-3	0	0	0	5	0	0	0	0	0	-7	0	0	0	0	0	0
$d_2^{(1)}(\ell)$	-4	0	0	0	0	9	0	0	0	0	0	0	-16	0	0	0	0
$d_3^{(1)}(\ell)$	-5	0	0	0	0	0	14	0	0	0	0	0	0	0	-30	0	0
$d_4^{(1)}(\ell)$	-6	0	0	0	0	0	0	20	0	0	0	0	0	0	0	0	-50
$d_5^{(1)}(\ell)$	-7	0	0	0	0	0	0	0	27	0	0	0	0	0	0	0	0
$d_6^{(1)}(\ell)$	-8	0	0	0	0	0	0	0	0	35	0	0	0	0	0	0	0
$d_7^{(1)}(\ell)$	-9	0	0	0	0	0	0	0	0	0	44	0	0	0	0	0	0
$d_8^{(1)}(\ell)$	-10	0	0	0	0	0	0	0	0	0	0	54	0	0	0	0	0
$d_9^{(1)}(\ell)$	-11	0	0	0	0	0	0	0	0	0	0	0	65	0	0	0	0
$d_{10}^{(1)}(\ell)$	-12	0	0	0	0	0	0	0	0	0	0	0	0	77	0	0	0
$d_{11}^{(1)}(\ell)$	-13	0	0	0	0	0	0	0	0	0	0	0	0	0	90	0	0
$d_{12}^{(1)}(\ell)$	-14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	104	0

The higher coefficients generically do not stabilize to constant numbers

$\frac{1}{16}$**-BPS index**

$$\frac{I_N(q)}{I_\infty(q)} = 1 + q^{N+1} \sum_{\ell=0}^{\infty} d_N^{(1)}(\ell)\, q^\ell$$

ℓ	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$d_1^{(1)}(\ell)$	-6	6	-3	-6	21	-36	27	30	-92	132	-90	-106	369	-444	164	486	-1221
$d_2^{(1)}(\ell)$	-10	12	-9	0	21	-54	83	-102	72	128	-459	744	-697	12	1440	-3240	4182
$d_3^{(1)}(\ell)$	-15	20	-18	12	10	-54	111	-190	279	-288	49	630	-1653	2790	-3303	1800	2938
$d_4^{(1)}(\ell)$	-21	30	-30	30	-12	-36	111	-234	417	-600	657	-480	-219	2118	-5256	8904	-11484
$d_5^{(1)}(\ell)$	-28	42	-45	54	-45	0	83	-234	486	-808	1113	-1368	1396	-642	-1665	6548	-14415
$d_6^{(1)}(\ell)$	-36	56	-63	84	-89	54	27	-190	486	-912	1417	-20[illegible]4	2688	-2942	2205	234	-5967
$d_7^{(1)}(\ell)$	-45	72	-84	120	-144	126	-57	-102	417	-912	1569	2478	3657	-4782	5430	-5178	2811

Formula for these coefficients?

cf [Imamura + Arai, Fujiwara, Mori,'19-'21]

[Gaiotto-Lee '21; Lee '22]

The giant graviton expansion

- Unitary matrix model $\mathbf{g} = (g_1, g_2, g_3, \dots)$

$$Z_N(\mathbf{g}) = \int_{U(N)} \mathrm{d}U \, \exp\left(\sum_{k=1}^{\infty} \frac{1}{k} g_k \operatorname{Tr} U^k \operatorname{Tr} U^{-k} \right)$$

- Free fermion ensemble $\mathbf{t} = (t_1, t_2, t_3, \dots)$

$$Z_N(\mathbf{g}) = \langle \langle \mathcal{O}_N \rangle \rangle_{\mathbf{g}} \equiv \int \mathrm{d}\mathbf{t} \, \mathrm{d}\bar{\mathbf{t}} \, \mathrm{e}^{-\mathbf{t}\bar{\mathbf{t}}/\mathbf{g}} \langle \bar{\mathbf{t}} \,|\, \mathcal{O}_N \,|\, \mathbf{t} \rangle$$

- Giant graviton expansion

$$\frac{Z_N(\mathbf{g})}{Z_\infty(\mathbf{g})} = \sum_{m=0}^{\infty} G_N^{(m)}(\mathbf{g}) \longrightarrow \mathrm{O}(\mathbf{g}^{mN}) \quad \left(\det(\cdots)_{N\times N}\right)^m$$

[S.M. ’22]

Derivation of the formula

1. Linearize the interactions using the Stratonovich-Hubbard trick

$$Z_N(\mathbf{g}) \;=\; \int_{U(N)} \mathrm{d}U \;\exp\left(\sum_{k=1}^{\infty} \frac{1}{k}\, g_k \,\mathrm{Tr}\, U^k \,\mathrm{Tr}\, U^{-k} \right)$$

$$\widetilde{Z}_N(\mathbf{t}^+, \mathbf{t}^-) \;=\; \int_{U(N)} \mathrm{d}U \;\exp\left(\sum_{k=1}^{\infty} \frac{1}{k} \left(t_k^+ \,\mathrm{Tr}\, U^k \;+ t_k^- \,\mathrm{Tr}\, U^{-k} \right) \right)$$

1. Linearize the interactions using the Stratonovich-Hubbard trick

$$Z_N(\mathbf{g}) \;=\; \int_{U(N)} \mathrm{d}U \;\exp\Bigg(\sum_{k=1}^{\infty} \frac{1}{k}\, g_k \,\mathrm{Tr}\, U^k \,\mathrm{Tr}\, U^{-k} \Bigg)$$

$$Z_N(\mathbf{g}) \;=\; \langle\, \widetilde{Z}_N \rangle_{\mathbf{g}}$$

$$\widetilde{Z}_N(\mathbf{t}^+, \mathbf{t}^-) \;=\; \int_{U(N)} \mathrm{d}U \;\exp\Bigg(\sum_{k=1}^{\infty} \frac{1}{k} \Big(t_k^+ \,\mathrm{Tr}\, U^k \;+ t_k^- \,\mathrm{Tr}\, U^{-k} \Big)\Bigg)$$

$$\langle\, f \,\rangle_{\mathbf{g}} := \prod_{k=1}^{\infty} \int \frac{\mathrm{d}t_k^+ \,\mathrm{d}t_k^-}{2\pi k g_k}\, \mathrm{e}^{-t_k^+ \, t_k^- / k g_k}\; f(\mathbf{t}^+, \mathbf{t}^-)$$

1. or, equivalently, linearize the sum of characters

[S.M. '22]

$$Z_N(\mathbf{g}) = \sum_{\ell(\mu)\leq N} \sum_{\lambda} \frac{\mathbf{g}^\lambda}{z_\lambda} \chi^\mu(\lambda)^2$$

$$Z_N(\mathbf{g}) = \langle \widetilde{Z}_N \rangle_{\mathbf{g}}$$

$$\widetilde{Z}_N(\mathbf{t}^+, \mathbf{t}^-) = \sum_{\ell(\mu)\leq N} S_\mu(\mathbf{t}^+)\, S_\mu(\mathbf{t}^-)$$

Partitions of integers

$$\lambda = 1^{r_1}\, 2^{r_2}\, \ldots$$

$$\mathbf{g}^\lambda = g_1^{r_1}\, g_2^{r_2}\, \ldots$$

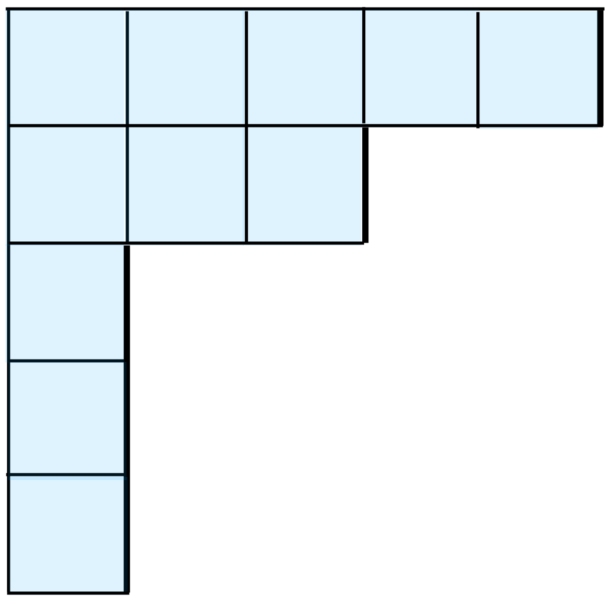
$$z_\lambda = \prod_{i\geq 1} r_i!\, i^{r_i}$$

Schur functions $S_\mu(\mathbf{t}) = \sum_{\lambda} \frac{\mathbf{t}^\lambda}{z_\lambda} \chi^\mu(\lambda)$

2. Sum over partitions = FF observable

[Okounkov ’99]

[cf Nekrasov-Okounkov ’03]



$$\lambda = (5,3,1,1,1)$$

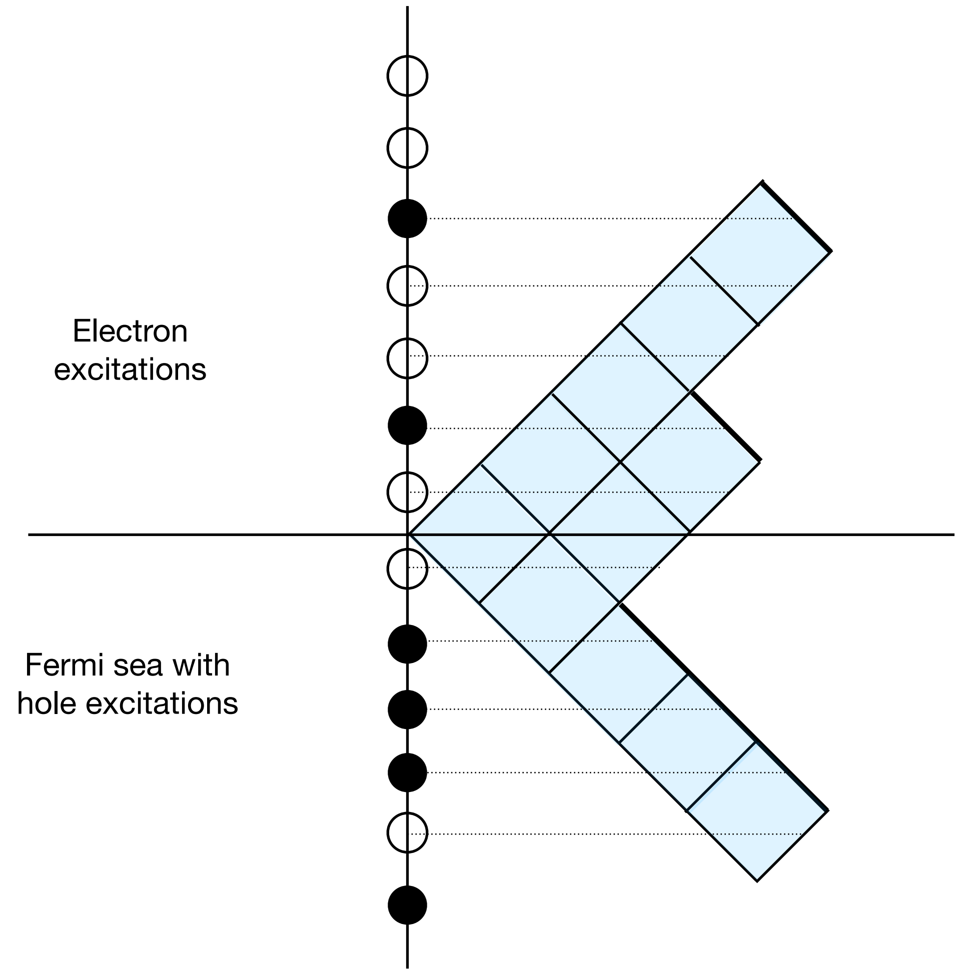
$$= 1^3\, 3^1\, 5^1$$

2. Sum over partitions = FF observable

[Okounkov '99]

[cf Nekrasov-Okounkov '03]

$$|\,\lambda\,\rangle = \psi_{-\frac{3}{2}}\,\psi_{-\frac{9}{2}}\,\overline{\psi}_{-\frac{1}{2}}\,\overline{\psi}_{-\frac{9}{2}}\,|\,0\,\rangle$$



$$\lambda = (5,3,1,1,1) = 1^3\,3^1\,5^1$$

- Schur basis $\quad |\,\mathbf{t}\,\rangle := \sum_{\mu} S_\mu(\mathbf{t})\,|\,\mu\,\rangle$

$$\Rightarrow \quad \langle\,\mathbf{t}^+\,|\,N_r\,|\,\mathbf{t}^-\,\rangle = \sum_{r\,\in\,|\,\mu\,\rangle} S_\mu(\mathbf{t}^+)\,S_\mu(\mathbf{t}^-)$$

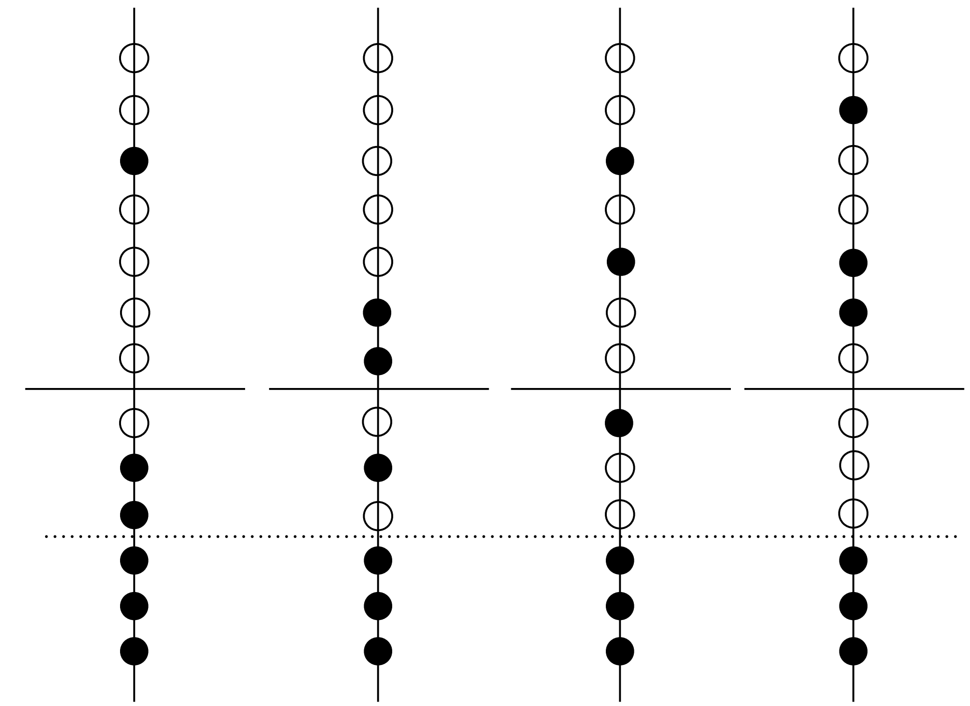
Excitation number

2. Sum over partitions = FF observable = sum over determinants

[Borodin-Okounkov '99]

$$\widetilde{Z}_N(\mathbf{t}^+, \mathbf{t}^-) \;=\; \sum_{\ell(\mu)\leq N} S_\mu(\mathbf{t}^+)\, S_\mu(\mathbf{t}^-)$$

$$= \langle\, \mathbf{t}^+ \,|\, \prod_{\substack{r<-N \\ r\,\in\,\mathbb{Z}+\frac{1}{2}}} (1 - N_r)\,|\, \mathbf{t}^- \,\rangle$$



Excite any number ≤ N (=3) of electrons from below the sea

$$\widetilde{Z}_N(\mathbf{t}^+, \mathbf{t}^-) \;=\; \sum_{m=0}^{\infty} (-1)^m \sum_{\substack{N<r_1<\cdots<r_m \\ r_i\in\mathbb{Z}+\frac{1}{2}}} \langle\, \mathbf{t}^+ \,|\, \psi_{r_1} \overline{\psi}_{-r_1} \ldots \psi_{r_m} \overline{\psi}_{-r_m} \,|\, \mathbf{t}^- \,\rangle$$

Fermionic determinant

2. Determinants can be calculated using bosonization

[Borodin-Okounkov '99]

$$\frac{\widetilde{Z}_N(\mathbf{t}^+,\mathbf{t}^-)}{\widetilde{Z}_\infty(\mathbf{t}^+,\mathbf{t}^-)} = \sum_{m=0}^{\infty} (-1)^m \sum_{\substack{N<r_1<\cdots<r_m \\ r_i \in \mathbb{Z}+\frac{1}{2}}} \det\Big(\widetilde{K}(r_i, r_j\,;\,\mathbf{t}^+,\mathbf{t}^-)\Big)_{i,j=1}^m$$

$$\sum_{r,s\,\in\,\mathbb{Z}+\frac{1}{2}} \widetilde{K}(r,s\,;\,\mathbf{t}^+,\mathbf{t}^-)\, z^r\, w^{-s} = \frac{J(z;\,\mathbf{t}^+,\mathbf{t}^-)}{J(w;\,\mathbf{t}^+,\mathbf{t}^-)}\,\frac{\sqrt{zw}}{z-w}$$

$$|w| < |z|$$

$$J(z;\,\mathbf{t}^+,\mathbf{t}^-) = \exp\left(\sum_{k=1}^{\infty} t_k^+\, z^k - \sum_{k=1}^{\infty} t_k^-\, z^{-k}\right)$$

3. The giant graviton expansion is obtained by transforming back to the original MM

[S.M. '22]

$$\frac{Z_N(\mathbf{g})}{Z_\infty(\mathbf{g})} = \sum_{m=0}^{\infty} G_N^{(m)}(\mathbf{g})$$

Contribution of m giants

$$G_N^{(m)}(\mathbf{g}) = (-1)^m \sum_{\substack{N<r_1<\cdots<r_m \\ r_i\in\mathbb{Z}+\frac{1}{2}}} \left\langle \frac{\widetilde{Z}_\infty}{Z_\infty} \det\big(\widetilde{K}(r_i,r_j)\big)_{i,j=1}^{m} \right\rangle_{\mathbf{g}}$$

- Ensemble average of fermionic $2m$-point function
- Energy at least mN

One-giant contribution has a suggestive formula in terms of dual single-letter trace

[S.M. '22]

$$\sum_{N\in\mathbb{Z}} G^{(1)}_{f,N}(q)\,\zeta^{-N} = \frac{1}{(1-\zeta)\,(1-1/\zeta)} \prod_{k=1}^{\infty}\left(\frac{(1-q^k)^2}{(1-q^k\,\zeta)\,(1-q^k/\zeta)}\right)^{\widehat{a}_k}$$

$|q| < |\zeta| < 1$

Dual single-letter trace

$$\frac{f(q)}{1-f(q)} = \widehat{f}(q) = \sum_{n=1}^{\infty} \widehat{a}_n\, q^n$$

Checks of the formula

Examples $\frac{1}{2}$-BPS index (Known analytic formula)

$$\frac{\mathcal{I}_N^{1/2}(q)}{\mathcal{I}_\infty^{1/2}(q)} = \sum_{m=0}^{\infty} \frac{(-1)^m q^{\binom{m+1}{2}}}{(q)_m} q^{mN}$$

$$(q)_m = \prod_{k=1}^{m}(1-q^k)$$

N=1

$$\frac{I_1^{1/2}(q)}{I_\infty^{1/2}(q)} - 1 = -q^2 - q^3 - q^4 + q^7 + q^8 + q^9 + q^{10} + q^{11} - q^{15} + \ldots$$

$$G^{(1)}_{\frac{1}{2},1}(q) = -q^2 - q^3 - q^4 - q^6 - q^8 - q^9 - q^{10} - 2q^{12} - q^{15} + \ldots$$

N=2

$$\frac{I_2^{1/2}(q)}{I_\infty^{1/2}(q)} - 1 = -q^3 - q^4 - q^5 - q^6 + q^9 + q^{10} + 2q^{11} + q^{12} + 2q^{13} + q^{14} + \ldots$$

$$G^{(1)}_{\frac{1}{2},2}(q) = -q^3 - q^4 - q^5 - q^6 - q^8 - q^{10} - 2q^{12} - q^{14} + \ldots$$

N=3

$$\frac{I_3^{1/2}(q)}{I_\infty^{1/2}(q)} - 1 = -q^4 - q^5 - q^6 - q^7 - q^8 + q^{11} + q^{12} + 2q^{13} + 2q^{14} + 2q^{15} + 2q^{16} + \ldots$$

$$G^{(1)}_{\frac{1}{2},3}(q) = -q^4 - q^5 - q^6 - q^7 - q^8 - q^{10} - q^{12} - q^{14} - q^{15} - q^{16} + \ldots$$

Examples $\frac{1}{16}$-BPS index (Numerical calculations)

N=1

$$\frac{I_1^{1/16}(q)}{I_\infty^{1/16}(q)} - 1 = -6q^4 + 6q^5 - 3q^6 - 6q^7 + 21q^8 - 36q^9 + 27q^{10} + 30q^{11} - 92q^{12} + 132q^{13} - 90q^{14} - 106q^{15} + 369q^{16} - 444q^{17} + \dots$$

$$G^{(1)}_{\frac{1}{16},1}(q) = -6q^4 + 6q^5 - 3q^6 - 6q^7 + 21q^8 - 36q^9 + 27q^{10} + 30q^{11} - 148q^{12} + 270q^{13} - 336q^{14} + 202q^{15} + 348q^{16} - 1392q^{17} + \dots$$

N=2

$$\frac{I_2^{1/16}(q)}{I_\infty^{1/16}(q)} - 1 = -10q^6 + 12q^7 - 9q^8 + 21q^{10} - 54q^{11} + 83q^{12} - 102q^{13} + 72q^{14} + 128q^{15} - 459q^{16} + 744q^{17} + \dots$$

$$G^{(1)}_{\frac{1}{16},2}(q) = -10q^6 + 12q^7 - 9q^8 + 21q^{10} - 54q^{11} + 83q^{12} - 102q^{13} + 72q^{14} + 128q^{15} - 585q^{16} + 1122q^{17} + \dots$$

N=3

$$\frac{I_2^{1/16}(q)}{I_\infty^{1/16}(q)} - 1 = -15q^8 + 20q^9 - 18q^{10} + 12q^{11} + 10q^{12} - 54q^{13} + 111q^{14} - 190q^{15} + 279q^{16} - 288q^{17} + 49q^{18} + 630q^{19} - 1653q^{20} + 2790q^{21} + \dots$$

$$G^{(1)}_{\frac{1}{16},3}(q) = -15q^8 + 20q^9 - 18q^{10} + 12q^{11} + 10q^{12} - 54q^{13} + 111q^{14} - 190q^{15} + 279q^{16} - 288q^{17} + 49q^{18} + 630q^{19} - 1905q^{20} + 3658q^{21} + \dots$$

Comments and speculations

- New index formula is like original SYM index.
  $m \leftrightarrow N$ duality?
- D-branes in AdS? $f \to \widehat{f}$ from curvature?
  [cf Imamura + Arai, Fujiwara, Mori, '19-'21]
- Formal framework?
  Open-closed-open string theory [cf Gopakumar '10]
  Double-scaled v/s Konstevich matrix models
  [Witten-Konstevich'91-'92]
- Integrability? Tau-functions?
- Convergent expansion $\Longrightarrow$ Giants form BHs!

Thank you very much!