(Exponential) Networks, A-branes and Enumerative invts.

Tuesday, April 5, 2022 1:02 AM

Based on 1811.02875 w/ S. Banerjee & P. Longhi
2201.12223

Outline : - Physics (where EN come from)
- E.N. (definition)
- Enumerative invts. from EN $\Omega([L])$ $L \in H_3(X_3, \mathbb{Z})$
- Interpretation of what $\Omega([L])$ is counting from an A-model p.o.v.

Physics M-theory on CY 3-folds.

Recall: $X_3 = \mu^{-1}(\vec{\zeta}_{FI}) / U(1)^k$ Toric CY 3-fold

M-theory on $\underbrace{S^1 \times \mathbb{R}^4}_{\text{5d SUSY th. (by geom. engineering)}} \times X_3 \leftarrow$

Goal: Count (5d) BPS corresponding to M5 and M2 (bound states) branes on
$\rightarrow \begin{cases} M2 \text{ on } \mathbb{R} \times \mathcal{C}_2 \\ M5 \text{ on } S^1 \times \mathbb{R} \times \mathcal{C}_4 \end{cases}$ $\mathcal{C}_a \subset X_3$ (compact) a-cycles

reducing to IIA, these are bound states of D0+D2+D4+D6 (DT invts.)

How to compute these invts using networks?
we introduce a defect M5 on $\underbrace{\mathbb{R}^2 \times S^1 \times L_{AV}}_{\text{3d SUSY th.}}$, $L_{AV} \subset X_3$, L_{AV} is a Aganagic-Vafa Lagrangian in X_3
$(L_{AV} \simeq \mathbb{R}^2 \times S^1)$

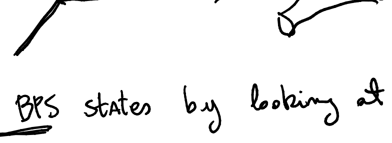
Considering the A-brane corresp. to L_{AV} i.e. $(L_{AV}, \nabla_{U(1)})$ flat conn.

$\text{Moduli}(L_{AV}, \nabla_{U(1)}) \cong \Sigma \subset (\mathbb{C}^*)^2$

Σ = mirror curve of X_3

(Hori-Vafa : $X_3^\vee = \{ uv - F(x,y) = 0 \mid u,v \in \mathbb{C},\ x,y \in \mathbb{C}^* \}$, $\Sigma := \{ F(x,y) = 0 \}$)

e.g. $\mathbb{C}^3$ |



$\cong S^2 \setminus \{3 \text{ pts.}\} = \Sigma_{\mathbb{C}^3}$

Idea : count 5d BPS states by looking at 3d-5d BPS states. $\leftarrow$
(2d-4d BPS states for SN)

Exponential Networks

$\Sigma \subset \mathbb{C}^* \times \mathbb{C}^*$ $x = e^X$ \ log var. $\lambda = Y dX$

Exponential Networks

$\Sigma \subset \mathbb{C}^*_x \times \mathbb{C}^*_y$ $\qquad$ $x = e^X$, $y = e^Y$ } log. vars , $\underline{\underline{\lambda = Y\,dX}}$

$F(x,y)=0$ $\qquad$ $(x,y) \in \mathbb{C}^*_x \times \mathbb{C}^*_y$

Laurent poly. in x, y

$$\tilde{\Sigma} \xrightarrow{\tilde{\pi}} \underbrace{\Sigma \xrightarrow{\pi'} \mathbb{C}^*_x}_{\Sigma \text{ as a branched cover of } \mathbb{C}^*_x}$$

$(x, Y_j + 2\pi i N) \in \tilde{\Sigma}$ ← log. cover $\qquad$ $(x, y_i(x)) \in \Sigma$ $\qquad$ $x \in \mathbb{C}^*_x$ $\qquad$ i = branches of π'

$$\boxed{F(x, y(x)) = 0}$$

$N \in \mathbb{Z}$ - branches of $\tilde{\pi}$

E.N is made of sols. of a differential eq., for fixed $\theta \in [0, 2\pi)$, τ - real param.

$m \in \mathbb{Z}$

$$\boxed{\left(Y_j(x) - Y_i(x) + 2\pi i m\right) \frac{dX(\tau)}{d\tau} = e^{i\theta}}$$

$i = \sqrt{-1}$

(A curve in $\tilde{\Sigma}$)

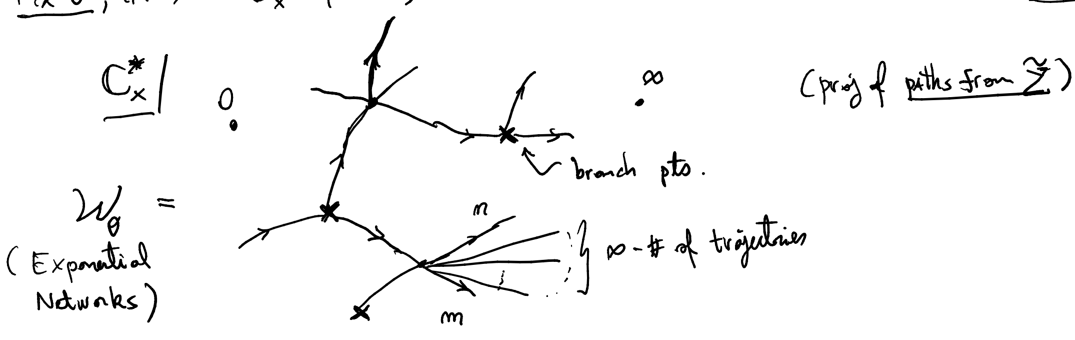
w/ bdry. cdts.

i) trajectories start (at $\tau \to -\infty$) at branch pts. of π' (i.e. where $y_j(x_B) = y_i(x_B)$) [x_B: branch pt. of π']

w/ $m = 0$

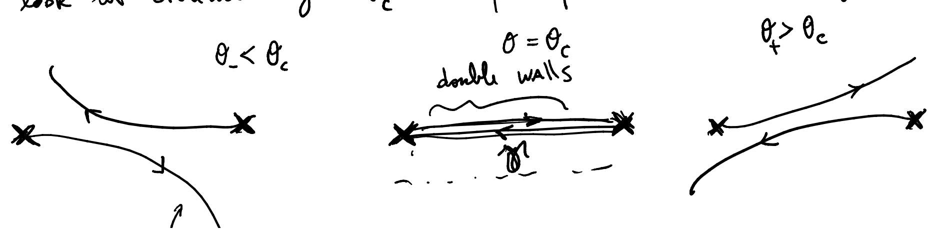
ii) trajectories start at intersections of other trajectories (they can have $m \neq 0$, but need to satisfy some consistency cdts.)

Fix θ, in the $\mathbb{C}^*_x$ plane; we can project these trajectories and get (assume all branch pts are simple)



We can extract BPS invts from $\mathcal{W}_\theta$

we need to look at critical angles θ_c → (part of) the network degenerates.



$\mathcal{E}$-walls

a network $\mathcal{W}_\theta$ gives you a nonabelianization map. i.e. an isomorphism between a flat $GL(1)$ connection on $\widetilde{\Sigma}$ and a $GL(\infty)$ flat conn. on $\mathbb{C}^*_x$ (i.e. $\underset{\mathbb{C}^*_x}{\overset{E}{\downarrow}} \cong \bigoplus_{j}\bigoplus_{N} \mathcal{L}_{j,N}$) — j: sheets of π', N: sheets of $\widetilde{\pi}$, $\mathcal{L}_{j,N}$: Line bundles

This isomorphism is not well defined at $\theta = \theta_c$ and it jumps from θ_- to θ_+

$\wp \subset \mathbb{C}^*_x$ a path

$$\mathrm{Hol}_\wp(\nabla^{ma}) = \underline{F(\wp,\theta_+)}$$

∇^{ma}: $GL(\infty)$ conn. ; $F(\wp,\theta_+)$: fn of $\underline{X_a := \mathrm{Hol}_a(\nabla^{ab})}$, a - paths on $\widetilde{\Sigma}$

$$F(\wp,\theta_+) = \mathcal{K}(F(\wp,\theta_-)) \text{ for any } \wp$$

$\mathcal{K}$ is a map that acts on X_a's as $\mathcal{K}(X_a) = X_a \cdot \prod_{k=1}^{\infty}(1 + X_{k\gamma})^{\overbrace{\langle L(k\gamma), a\rangle}^{\in \mathbb{Z}}}$ — int. between a lift $L(k\gamma)$ of $k\gamma$ to $\widetilde{\Sigma}$ and a

$$X_a \cdot X_b = \begin{cases} X_{ab} & ab\text{ - concat. if } \mathrm{end}(a) = \mathrm{beg}(b) \\ 0 & \sim \end{cases}$$

γ is the closed cycle in Σ determined by the degeneration

$$L(k\gamma) = \underbrace{\Omega(k\gamma)}_{\in \mathbb{Z}}\,\underbrace{\widetilde{\pi}^{-1}(k\gamma)}_{\text{cycle in } \widetilde{\Sigma}}$$

$k\gamma \xrightarrow{(\mathrm{Mir})}$ bound state of B-branes on X_3

$\widetilde{\Sigma} \to \Sigma$ - $\underset{\text{Curve}}{\mathrm{Mir}}(X_3)$

are the 5d BPS degeneracies.

1-cycle in Σ $\rightsquigarrow L_\gamma^{\text{sLag}}$ in $X_3^\vee$

Geom. int. of $\Omega(\gamma)$ (for primitive cycles $\gamma \in H_1(\Sigma, \mathbb{Z})$)

Classically if $L \subset X_3^\vee = \{uv - F(x,y) = 0\}$ is a special lagrangian of class $[L] \in H_3(X_3^\vee, \mathbb{Z})$

$\mathcal{M}$:= moduli of defs. of L inside $X_3^\vee$

$\nabla_{U(1)}$ flat conn. on L

$$\boxed{\begin{array}{c} T^{b_1(L)} \longrightarrow \mathcal{M} := \mathrm{Moduli}(L, \nabla_{U(1)}) \\ \downarrow \\ \mathcal{M} \end{array}}$$

$b_1(L) = \dim(H_1(L,\mathbb{Z}))$

McLean $\dim_{\mathbb{R}} \mathcal{M} = b_1(L)$

bb

$\Rightarrow \dim_{\mathbb{C}} \mathcal{M} = b_1(L)$

each pt. in $\mathcal{M}$ corresponds to a, possibly singular, special Lag.

when 1-cycles of L pinches

$\mathcal{D} \subset \mathcal{M}$ is the subset of points in $\mathcal{M}$, where L is maximally degenerate

(Main) Claim: $\Omega([L_\gamma]) = (-1)^{b_1(L)} \cdot |\mathcal{D}| = (-1)^{b_1(L)} \underbrace{\chi(\mathcal{M})}$

$\Omega([L_\gamma]) = \Omega(\gamma)$, $\quad \chi(\mathcal{M}) = \sum'(\text{fixed pts of } (\mathbb{C}^*)^{b_1(L)} \text{ action on } \mathcal{M})$

Connection w/ EN (or Spectral Netw.): * double walls corresponds to maximally degenerate lagrangians in $X_3^\vee$

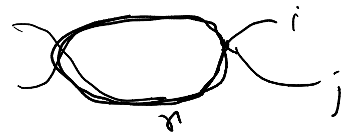
* Moreover these degenerate lags., corresponds to leaves of a foliation determined by the differential λ (or λ_{SW})

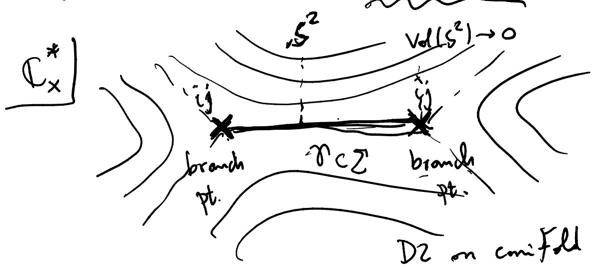
Checks: - single D2 in $X_3 = \mathcal{O}(-1)^2 \to \mathbb{P}^1$ or $\mathcal{O}\oplus\mathcal{O}(-2) \to \mathbb{P}^1$

- spectral networks : A_N-type

- — " — : N-herd states

$\Sigma =$ 



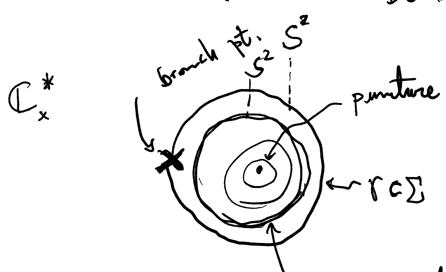


$\gamma \rightsquigarrow L_\gamma \subset X_3^\vee \quad \to S^3 \simeq L_\gamma$

S^2-fibration over the segment on $\mathbb{C}^*_x$

$b_1(L) = 0$

$\Omega(L_\gamma) = 1$



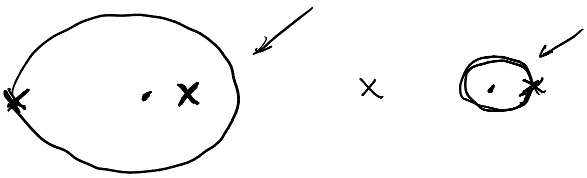
critical / maximally deg. Lag.

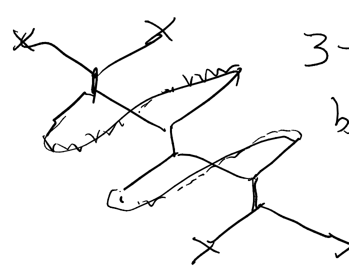
$L_\gamma \simeq S^3/\sim$ two points identified

$\Omega(L_\gamma) = -1$

generic leaf $b_1(L) = 1$

gen. $L \simeq S^2 \times S^1$





3-herd

$b_1 = 2$