Quantum Spectral Curve and AdS$_3$/CFT$_2$

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arXiv:2109.06164 w D. Volin

arXiv:2211.07810 w A. Cavaglià, N. Gromov and P. Ryan

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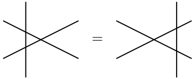
Introduction and setting the stage

- The last two decades has seen an immense progress in solving $D = 4$ planar $\mathcal{N} = 4$ SYM.

- Underlying reason: Integrability



[Minahan,Zarembo,'02]
[Metsaev, Tseytlin,'02]
[+ A lot of work]

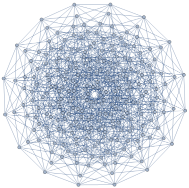
- Mayor accomplishment: The spectral problem was solved!

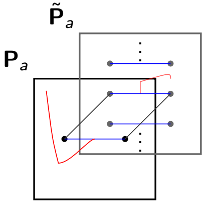
$$\left\langle \mathcal{O}(x)\bar{\mathcal{O}}(y)\right\rangle \sim \frac{1}{|x-y|^{2\Delta}} \xleftarrow{\text{Computable!}}$$

- Most efficient formulation? [Gromov, Kazakov,Leurent,Volin '13'14]

> The Quantum Spectral Curve

$$\tilde{\mathbf{P}}_a = \mu_{ab}\mathbf{P}^b$$





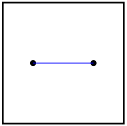
What is the $\mathcal{N}=4$ QSC?

- The QSC is a set of 256 Q-functions, they depend on 1 complex parameter: u. The simplest Q-functions are called $\mathbf{P}_a, \mathbf{P}^a$

$\mathbf{P}_a(u)$

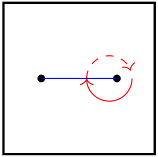
Spectral parameter

$\mathbf{P}_a, \mathbf{P}^a$

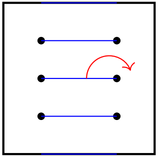


- Analytic continuation is controlled by the so-called $\mathbf{P}\mu$ system:

$\mathbf{P}_a$



$\tilde{\mathbf{P}}_a = \mu_{ab}\mathbf{P}^b$



$$\tilde{\mathbf{P}}_a = \mu_{ab}\mathbf{P}^b\,,$$
$$\tilde{\mu}_{ab} - \mu_{ab} = \mathbf{P}_a\tilde{\mathbf{P}}_b - \tilde{\mathbf{P}}_a\mathbf{P}_b\,.$$

- The asymptotics encodes quantum numbers, in particular the conformal dimension!

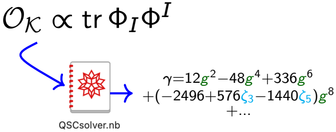
$$\mu_{12} \sim_{u\to\infty} u^{\Delta - J_1}$$

- For reviews, see [Gromov '17;Kazakov'18,Levkovich-Maslyuk '19].

Elevator pitch for $\mathcal{N}=4$ QSC

- Analytic weak coupling computations available ("Black box")

  [Marboe,Volin 18']

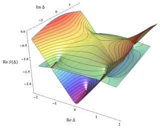


$\mathcal{O}_{\mathcal{K}} \propto \mathrm{tr}\,\Phi_I\Phi^I$

$$\gamma = 12g^2 - 48g^4 + 336g^6 + (-2496 + 576\zeta_3 - 1440\zeta_5)g^8 + \ldots$$

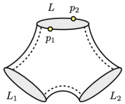
- Analytic continuation in spin

  [Gromov,Levkovich-Maslyuk,Sizov '15]



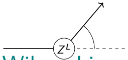
- Structure constants

  [Basso, Georgoudis, Klemenchuk Sueiro '22]



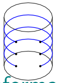
- There also exists many exciting variations and deformations:

[Gromov,Levkovich-Maslyuk '15]



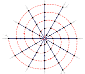
Wilson Lines

[Klabbers,van Tongeren '17]



η- deformations

[Gromov et al '17]



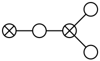
Fishnets

What I will talk about

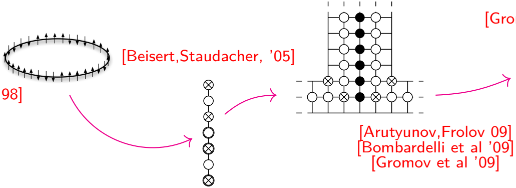
- The success of QSC in $\mathcal{N} = 4$ raises the question: Can we extend it to other theories?
- Yes! Already successfully done for
  - $\mathrm{AdS}_4/\mathrm{CFT}_3$ (ABJM) ✓

    [Bombardelli, Cavaglià,Fioravanti,Gromov,Tateo '17]
  - The Hubbard Model ✓

    [Cavaglià, Cornagliotto,Mattelliano,Tateo '15]



- Today: Attempt to extend to $\mathrm{AdS}_3/\mathrm{CFT}_2$!
- Why is this exciting?
  - First attempt to investigate an "unknown" CFT using QSC.
  - First attempt to bootstrap a consistent QSC and avoid the long historical route of $\mathcal{N} = 4$.



[Braun,Derkachov,Manashov '98]
[Minahan,Zarembo, '02]

[Beisert,Staudacher, '05]

[Arutyunov,Frolov 09]
[Bombardelli et al '09]
[Gromov et al '09]

[Gromov et al '13]

QSC

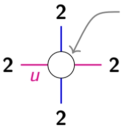
Plan of the talk

1. Crash course on Q-systems
2. QSC generalities, the case of $\mathcal{N} = 4$
3. Monodromy Bootstrap and $\mathrm{AdS}_3/\mathrm{CFT}_2$ conjecture.
4. Solving the curve.

Analytic Q-systems

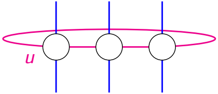
$\mathfrak{su}_2$ spin chain I

- Consider a homogeneous $\mathfrak{su}_2$ spin chain. This model has an R-matrix from which we can build a transfer-matrix



$$R(u) \propto \underbrace{(u - \mathtt{i})\mathbb{P}_{\mathsf{Sym}} + (u + \mathtt{i})\,\mathbb{P}_{\mathsf{ASym}}}_{\text{Polynomial}}$$

$t(u) =$



- The eigenvalues of $t(u)$: (Dressed Vacuum Form)

$$t(u) = (u - \tfrac{\mathtt{i}}{2})^L \frac{Q_1^{[2]}}{Q_1} + (u + \tfrac{\mathtt{i}}{2})^L \frac{Q_1^{[-2]}}{Q_1}$$

Q-function!

where $f^{[n]} = f(u + \frac{\mathtt{i}}{2} n)$.

- Q_1 is a polynomial $Q_1 = \prod_{i=1}^{M}(u - u_i)$.
  - Asymptotic of Q_1 encodes the quantum number M.
  - Polynomiality of $t(u)$ implies Bethe equations

$$\left.\frac{Q_1^{[2]}}{Q_1^{[-2]}}\right|_{Q_1=0} = -\left.\left(\frac{u+\frac{\mathtt{i}}{2}}{u-\frac{\mathtt{i}}{2}}\right)^L\right|_{Q_1} = 0$$

$\mathfrak{su}_2$ spin chain II

- Can introduce polynomial Q_2 and write $t(u)$ in polynomial form:

  $$t(u) = Q_1^{[2]} Q_2^{[-2]} - Q_1^{[-2]} Q_2^{[2]}$$

  Q_1, Q_2 must satisfy the QQ/Wronskian-relation

  $$\boxed{Q_1^+ Q_2^- - Q_1^- Q_2^+ = Q_{\bar{\emptyset}} \equiv u^L \,.}$$

- Symmetries of the system:
  - Gauge-transformations

    $$Q_a \to r\, Q_a, \qquad Q_{\bar{\emptyset}} \to r^+ r^- \, Q_{\bar{\emptyset}}$$

  - H-rotations

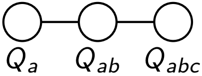
    $$Q_a \to H_a{}^b Q_b \,, \qquad (H^+)_a{}^b = (H^-)_a{}^b$$

- Benefits of QQ-system:
  - More efficient than Bethe equations.
  - Correctly deals with exceptional solutions.

$\mathfrak{su}_N$ Q-systems I

- To go to $\mathfrak{su}_n$ attach a Q-"vector" to nodes on the Dynkin diagram

A_3:



where $a = 1, \dots, n$.

- The various Q-functions are related by functional equations: QQ-relations:

Source term

$$Q^+_{Aa}Q^-_{Ab} - Q^-_{Aa}Q^+_{Ab} = Q_{Aab}Q_A \,, \qquad Q_{\bar{\emptyset}} = u^L \,.$$

$\bar{\emptyset} = 1234$

- QQ-relations leads to Nested Bethe Equations

$$\underbrace{\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}}_{A_3 \text{ Cartan Matrix}}$$

$$\left.\frac{Q_1^{[2]}}{Q_1^{[-2]}}\frac{Q_{12}^{[-1]}}{Q_1^{[+1]}}\right|_{Q_1=0} = -1$$

$$\left.\frac{Q_{12}^{[2]}}{Q_{12}^{[-2]}}\frac{Q_1^{[-1]}Q_{123}^{[-1]}}{Q_1^{[+1]}Q_{123}^{[+1]}}\right|_{Q_{12}=0} = -1$$

$$\left.\frac{Q_{123}^{[2]}}{Q_{123}^{[-2]}}\frac{Q_{12}^{[-1]}}{Q_{12}^{[+1]}}\right|_{Q_{123}=0} = -\left.\left(\frac{u+\frac{i}{2}}{u-\frac{i}{2}}\right)^L\right|_{Q_1=0}$$

$\mathfrak{su}_N$ Q-systems II

- Symmetries of QQ-relations?

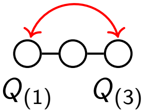
$$Q^+_{Aa}Q^-_{Ab} - Q^-_{Aa}Q^+_{Ab} = Q_{Aab}Q_A$$

- We still have gauge-transformations and rotations

$$Q_a \to r\,Q_a\,, \qquad Q_a \to H_a{}^b\,Q_b\,, \qquad (H^+)_a{}^b = (H^-)_a{}^b\,, \tag{1.1}$$

but also Hodge Duality

$$(Q_a)^\star \to Q^a \propto \epsilon^{abcd}Q_{bcd}$$



- Q-systems are very general
Change source terms $\implies$ change representations
Change analytic properties $\implies$ change integrable model

Summarizing $\mathfrak{su}_n$ Q-systems

- A $\mathfrak{su}_N$ Q-system consists of functions $Q_a, Q_{ab}, Q_{abc}, \ldots$ satisfying QQ-relations

$$Q^+_{Aa}Q^-_{Ab} - Q^-_{Aa}Q^+_{Ab} = Q_{Aab}Q_A\,.$$

- The Q-functions transform under gauge-transformations, rotations and Hodge duality.
- Philosophy from now on: Forget R-matrices, T,Y-functions etc and trust the Q-system.
- To get to QSC we need to generalize two aspects of the $\mathfrak{su}_N$ Q-system:
  - Supersymmetric Q-system
  - Analytic properties beyond polynomiality.
- Q-systems for arbitrary (super-) Lie algebras are still an active research direction [Mukhin,Varchenko '05, Masoero Raimondo Valeri '15-18, Koroteev,Zeitlin '18-21,Ferrando,Frassek,Kazakov,'20;SE,Shu,Volin'20 . . .]

Supersymmetric Q-systems I

- Now: Supersymmetric Q-systems! For simplicity: $\mathfrak{su}_{n|n}$.
- Introduce two Q-systems: $Q_{a|\emptyset}, Q_{\emptyset|i}$, $a, i = 1, 2, \ldots, n$.
- Connect through new functions $Q_{a|i}, Q_{\bar{\emptyset}|\bar{\emptyset}}$ that satisfy

$$Q_{\emptyset|i} \propto \frac{Q^{\pm}_{a|i}}{Q^{\pm}_{\bar{\emptyset}|\bar{\emptyset}}} Q^{a|\emptyset} , \quad Q_{a|\emptyset} \propto \frac{Q^{+}_{a|i}}{Q^{+}_{\bar{\emptyset}|\bar{\emptyset}}} Q^{\emptyset|i} , \quad Q^{+}_{a|i} - Q^{-}_{a|i} = Q_{a|\emptyset} Q_{\emptyset|i} \,.$$

- We can then build functions $Q_{A|I}$ from QQ-relations.

$$Q^{+}_{Aa|I} Q^{-}_{Ab|I} - Q^{-}_{Aa|I} Q^{+}_{Ab|I} = Q_{Aab|I} Q_{A|I}$$
$$Q^{+}_{A|Ii} Q^{-}_{A|Ij} - Q^{-}_{A|Ii} Q^{+}_{A|Ij} = Q_{A|Ii} Q_{A|Ij}$$
$$Q^{+}_{Aa|Ii} Q^{-}_{A|I} - Q^{-}_{Aa|Ii} Q^{+}_{A|I} = Q_{Aa|I} Q_{A|Ii}$$

Supersymmetric Q-systems II

- The supersymmetric QQ-relations also implies supersymmetric Nested Bethe Equations:

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\left.\frac{Q_{1|1}^{[+1]}}{Q_{1|1}^{[-1]}}\right|_{Q_{1|\emptyset}=0} = 1$$

$$\left.\frac{Q_{1|1}^{[2]} Q_{1|\emptyset}^{[-1]} Q_{1|12}^{[-1]}}{Q_{1|1}^{[-2]} Q_{1|\emptyset}^{[1]} Q_{1|12}^{[1]}}\right|_{Q_{1|1}=0} = -1$$

$$\left.\frac{Q_{1|1}^{[+1]}}{Q_{1|1}^{[-1]}}\right|_{Q_{1|12}=0} = \left(\frac{u+\frac{i}{2}}{u-\frac{i}{2}}\right)^{L}$$

- Once again there are symmetry transformations
  - Rotations

$$Q_{a|\emptyset} \to (H_{\mathrm{B}})_a{}^b Q_{b|\emptyset}\,, \qquad Q_{\emptyset|i} \to (H_{\mathrm{F}})_i{}^j Q_{\emptyset|j}\,,$$

  - Gauge-transformations

$$Q_{a|\emptyset} \to r_{\mathrm{B}} Q_{a|\emptyset}\,, \qquad Q_{\emptyset|i} \to r_{\mathrm{F}} Q_{\emptyset|i}\,,$$

  - Hodge

$$Q_{a|\emptyset} \to Q^{a|\emptyset} \propto \epsilon^{aA} Q_{A|\bar{\emptyset}}\,, \qquad Q_{\emptyset|i} \to Q^{\emptyset|i} \propto \epsilon^{iI} Q_{\bar{\emptyset}|I}\,,$$

Summary analytic Q-systems

- A $\mathfrak{su}_{n|n}$ Q-system is built from functions $Q_{A|I}$. They satisfy QQ-relations

$$\begin{aligned}
Q^+_{Aa|I}Q^-_{Ab|I} - Q^-_{Aa|I}Q^+_{Ab|I} &= Q_{Aab|I}Q_{A|I} \\
Q^+_{A|Ii}Q^-_{A|Ij} - Q^-_{A|Ii}Q^+_{A|Ij} &= Q_{A|Ii}Q_{A|Ij} \\
Q^+_{Aa|Ii}Q^-_{A|I} - Q^-_{Aa|Ii}Q^+_{A|I} &= Q_{Aa|I}Q_{A|Ii}
\end{aligned}$$

- The QQ-relations encodes Nested Bethe Equations.
- We can transform Q-functions using gauge-transformations,rotations and Hodge.
- We are now ready to go to QSC!

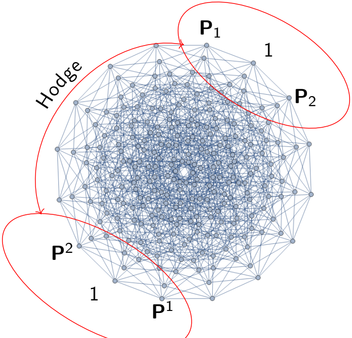
Quantum spectral curve for AdS$_5$/CFT$_4$

Algebraic aspects of AdS$_5$/CFT$_4$

- The underlying algebra for AdS$_5\times$ S^5 is $\mathfrak{psu}_{2,2|4}$. As we have seen basic Q-functions comes in two flavours $\underbrace{Q_{a|\emptyset}=\mathbf{P}_a}_{\text{compact}}$, $\underbrace{Q_{\emptyset|i}=\mathbf{Q}_i}_{\text{non-compact}}$.
- For the AdS$_5$/CFT$_4$ QSC they are related by

$$\mathbf{P}_a = -Q^{\pm}_{a|i}\mathbf{Q}^i\,, \qquad Q^{+}_{a|i}-Q^{-}_{a|i}=\mathbf{P}_a\mathbf{Q}_i\,, \qquad \underbrace{\mathbf{P}_a\mathbf{P}^a=0}_{\mathfrak{psu}_{2,2|4}}\,.$$

- Sometimes these functions are depicted on a Hasse diagram.

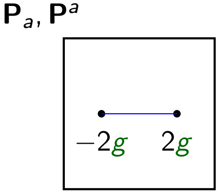


$$\mathbf{P}_a \simeq_{u\to\infty} A_a u^{M_a}$$
$$\mathbf{Q}_i \simeq_{u\to\infty} B_i u^{\hat{M}_i}$$

$$\Delta = \Delta^{(0)} + \gamma$$

Quantum Spectral Curve of AdS_5/CFT_4 I

- Simplest objects of QSC: $\mathbf{P}_a, \mathbf{P}^a$



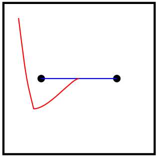
$g = \frac{\sqrt{\lambda}}{4\pi}$

't Hooft coupling

Quantum Spectral Curve of AdS_5/CFT_4 I

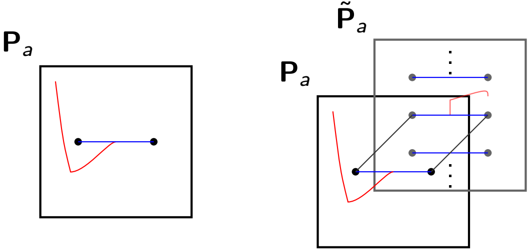
- Simplest objects of QSC: $\mathbf{P}_a, \mathbf{P}^a$

$\mathbf{P}_a, \mathbf{P}^a$



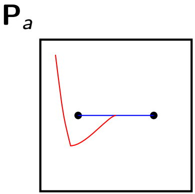
Quantum Spectral Curve of $\mathrm{AdS}_5/\mathrm{CFT}_4$ I

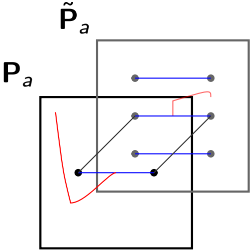
- Simplest objects of QSC: $\mathbf{P}_a, \mathbf{P}^a$:



Quantum Spectral Curve of AdS_5/CFT_4 I

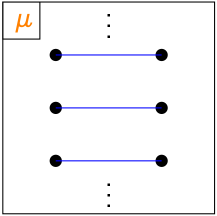
- Simplest objects of QSC: $\mathbf{P}_a(u)$:





- Continuation is under control:

$$\widetilde{\mathbf{P}}_a = \overset{\text{Rotation}}{\mu_{ab}}\underset{\text{Hodge Dual}}{\mathbf{P}^b}\,.$$



Quantum Spectral Curve of $\mathrm{AdS}_5/\mathrm{CFT}_4$ I

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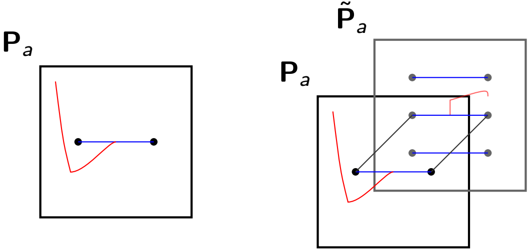

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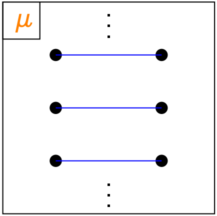


- How do we fix μ_{ab}?

The $\mathbf{P}\mu$-system

- The discontinuity of μ_{ab} on the cut on the real axis is computed as:

$$\tilde{\mu}_{ab} - \mu_{ab} = \mathbf{P}_a \tilde{\mathbf{P}}_b - \tilde{\mathbf{P}}_a \mathbf{P}_b \,,$$

- The full μ_{ab} can then be restored by demanding anti-symmetry, $\mathrm{Pf}(\mu_{ab}) = 1$ and mirror periodicity

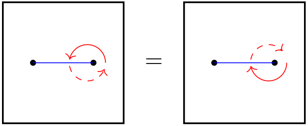
$$\tilde{\mu}_{ab} = \mu_{ab}^{[2]} \,.$$

- Supplementing these equations with

$$\tilde{\mathbf{P}}_a = \mu_{ab} \mathbf{P}^b$$

  gives the the $\mathbf{P}\mu$ system.

- A consequence of the $\mathbf{P}\mu$-system is that all cuts are of quadratic. I.e



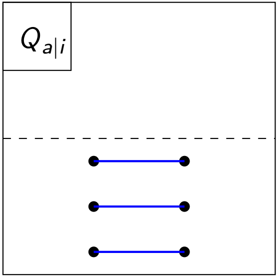
The remaining functions I

- We cannot demand that all Q-functions, $\mathbf{Q}_i, Q_{a,i}, \ldots$ only have a short-cut. But we can demand that the full Q-system is analytic in the upper half-plane.
- Lets look at $Q_{a|i}$, it satisfies:

$$Q^+_{a|i} - Q^-_{a|i} = -\mathbf{P}_a \mathbf{P}^b Q^+_{b|i}$$

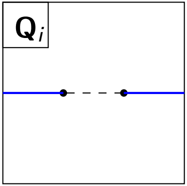
$$Q_{a|i}(u - \tfrac{i}{2}) = \left(\delta^b_a + \mathbf{P}_a(u)\mathbf{P}^b(u)\right) Q_{b|i}(u + \tfrac{i}{2})$$

- Giving an analytic structure



The remaining functions II

- What about $\mathbf{Q}_i$? From: $\mathbf{Q}_i = -Q^{\pm}_{a|i}\mathbf{P}^a$ and $\mathbf{P}\mu$-system it is possible to deduce that $\mathbf{Q}_i$ is a long-cut function.



- Furthermore, applying $Q^-_{a|i}$ leads to the $\mathbf{Q}\omega$-system

$$\tilde{\mathbf{Q}}_i = \omega_{ij}\mathbf{Q}^j \qquad \tilde{\omega}_{ij} - \omega_{ij} = \mathbf{Q}_i\tilde{\mathbf{Q}}_j - \tilde{\mathbf{Q}}_i\mathbf{Q}_j$$

where $\omega^{[2]} = \omega$ and is related to μ as

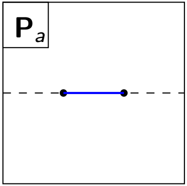
$$\omega_{ij} = Q^-_{a|i}\mu^{ab}Q^-_{b|j}\,.$$

Summary of AdS$_5$/CFT$_4$

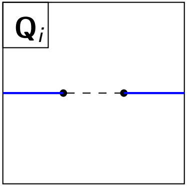
- The AdS$_5$/CFT$_4$ QSC is an analytic $\mathfrak{psu}(2,2|4)$ Q-system.
- The basic Q-functions are $\mathbf{P}_a, \mathbf{Q}_i$. The asymptotics of these functions encodes quantum numbers.
- One way of presenting the QSC is through the $\mathbf{P}\mu$ system

$$\tilde{\mathbf{P}}_a = \overset{\text{Rotations}}{\mu_{ab}}\underset{\text{Hodge Dual}}{\mathbf{P}^b}\,. \qquad \tilde{\mu}_{ab} - \mu_{ab} = \mathbf{P}_a\tilde{\mathbf{P}}_b - \tilde{\mathbf{P}}_a\mathbf{P}_b\,.$$

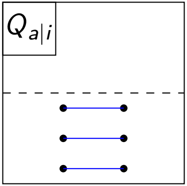
- The most important cut-structures



Short cut



Long cut

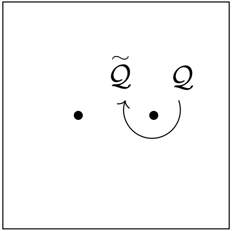


Ladder of cuts

Extending the QSC (Monodromy Bootstrap for $\mathfrak{su}_{2|2}$)

Monodromy Bootstrap

- Want to study QSC with symmetry group $\mathfrak{su}_{2|2}$ (Really $\mathfrak{gl}_{2|2}$).
- Informal idea: Monodromy around branch-point leads to a symmetry transformation of the Q-system

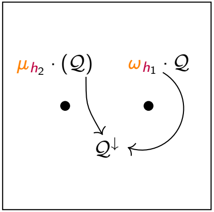


$$\underbrace{\tilde{Q} = S \cdot Q}_{\text{"Crossing"}}$$

- The precise way to proceed is given in [SE,Volin '21].

Monodromy bootstrap, technical details

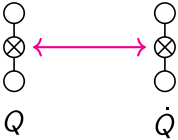
- Property 1: There exist a Q-system, $\mathcal{Q}$, analytical in the upper half-plane.
- Property 2: $\mathbf{P}_a$ has a short-cut, $\mathbf{Q}_i$ has a long-cut.
- Property 3: There $\underbrace{\text{exist a symmetry transformations}}_{\text{gauge+rotations+Hodge Duality}}$ sending $\mathcal{Q}$ to a lower half-plane analytic Q-system $\mathcal{Q}^{\downarrow}$.
- The symmetry transformation depends on the path



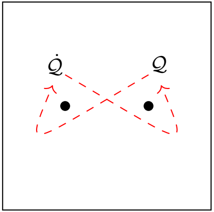
Gluing two Q-systens

- Following these rules a classification of models was obtained [SE,Volin '21].
- Today: Focus only on the case that might be relevant for $\mathrm{AdS}_3/\mathrm{CFT}_2$.
- Basic intuition: Need two UHPA Q-systems $Q, \dot{Q}$ to describe "left-and right-movers".

$$\underbrace{\mathfrak{psu}_{1,1|2}\oplus\mathfrak{psu}_{1,1|2}}_{\text{Q-system}}$$



- Connect them through the Monodromy Bootstrap procedure



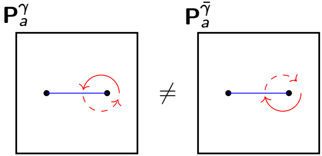
The Pμ-system and Qω-system

- Working out the details one finds the following **P**μ-system

$$\mathbf{P}_a^{\bar\gamma} = \mathbf{P}_{\dot b}\mu^{\dot b}{}_a\,, \quad (\mu_a{}^{\dot b})^{\gamma} - \mu_a{}^{\dot b} = \mathbf{P}_a(\mathbf{P}^{\dot b})^{\bar\gamma} - (\mathbf{P}_a)^{\gamma}\mathbf{P}^{\dot b}\,.$$

different Q-systems!

- Cuts are no longer quadratic!



- The **Q**ω-system is

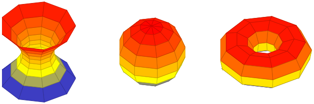
Opposite direction

$$\mathbf{Q}_k^{\gamma} = \omega_k{}^i\mathbf{Q}_i\,, \quad (\omega_k{}^i)^{\bar\gamma} - \omega_k{}^i = \mathbf{Q}_k(\mathbf{Q}^i)^{\bar\gamma} - (\mathbf{Q}_k)^{\gamma}\mathbf{Q}^i\,.$$

The $\mathrm{AdS}_3/\mathrm{CFT}_2$ Conjecture

- Conjecture: These equations describe the spectrum of planar string theory on $\mathrm{AdS}_3\times S^3\times T^4$ with pure RR-flux!

  [SE,Volin '21][Cavaglià,Gromov,Stefanski,Torrielli, 21']



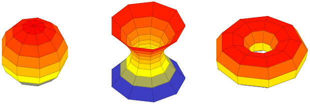
- What will we compute?
  - Large volume limit, reproduce the Asymptotic Bethe Ansatz.
  - Perturbative calculations for small length operators.

The $\text{AdS}_3/\text{CFT}_2$ Quantum Spectral Curve Conjecture

Quick Background on $\text{AdS}_3/\text{CFT}_2$ Integrability

- What do we hope to study? [Babichenko, Stefanski, Zarembo '09]

String Sigma Model on $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ with pure RR-flux



- What do we know about the system?
  - World-sheet S-matrix has been bootstrapped! Asymptotic Bethe equations have been obtained. Elementary excitations have dispersion relation:[Borsato, Ohlsson-Sax, Sfondrini, Stefanski '13,'14,'16]

$$E(p) = \sqrt{m^2 + 16g^2 \sin^2 \tfrac{p}{2}} \qquad\qquad m = \pm 1, 0$$

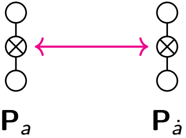
massless modes, new feature

  - Since recently there is is a TBA [Frolov,Sfondrini 21']
  - No easy checks at weak coupling (no data from dual CFT). (Some progress [Ohlsson-Sax,Sfondrini,Stefanski '14])

Large Volume Checks

- Recall the basic structure of the proposed QSC

$$\underbrace{\mathfrak{psu}_{1,1|2} \oplus \mathfrak{psu}_{1,1|2}}_{\text{Q-system}}$$



- S-matrix $\implies$ Asymptotic Bethe Equations. Try to match with exact BE in appropriate limit

$$\frac{Q^+_{1|1}}{Q^-_{1|1}} = 1$$

$$\frac{Q^{[2]}_{1|1}\mathbf{Q}_1^-(\mathbf{Q}^2)^-}{Q^{[-2]}_{1|1}\mathbf{Q}_1^+(\mathbf{Q}^2)^+} = -1$$

$$\frac{Q^+_{1|1}}{Q^-_{1|1}} = 1$$

$$\frac{Q^+_{\dot 1|\dot 1}}{Q^-_{\dot 1|\dot 1}} = 1$$

$$\frac{Q^{[2]}_{\dot 1|\dot 1}\mathbf{P}_{\dot 1}^-(\mathbf{P}^{\dot 2})^-}{Q^{[-2]}_{\dot 1|\dot 1}\mathbf{P}_{\dot 1}^+(\mathbf{P}^{\dot 2})^+} = -1$$

$$\frac{Q^+_{\dot 1|\dot 1}}{Q^-_{\dot 1|\dot 1}} = 1$$

$$1 = \prod_{j=1}^{K_2} \frac{y_{1,k} - x_j^+}{y_{1,k} - x_j^-} \prod_{j=1}^{K_{\bar 2}} \frac{1 - \frac{1}{y_{1,k}\bar x_j^-}}{1 - \frac{1}{y_{1,k}\bar x_j^+}}, \tag{4.16}$$

$$\begin{aligned}\left(\frac{x_k^+}{x_k^-}\right)^L = &\prod_{\substack{j=1\\ j\neq k}}^{K_2} \frac{x_k^+ - x_j^-}{x_k^- - x_j^+} \frac{1 - \frac{1}{x_k^+ x_j^-}}{1 - \frac{1}{x_k^- x_j^+}} \sigma^2(x_k, x_j) \prod_{j=1}^{K_1} \frac{x_k^- - y_{1,j}}{x_k^+ - y_{1,j}} \prod_{j=1}^{K_3} \frac{x_k^- - y_{3,j}}{x_k^+ - y_{3,j}} \\ &\times \prod_{j=1}^{K_{\bar 2}} \frac{1 - \frac{1}{x_k^+ \bar x_j^+}}{1 - \frac{1}{x_k^- \bar x_j^-}} \frac{1 - \frac{1}{x_k^+ \bar x_j^-}}{1 - \frac{1}{x_k^- \bar x_j^+}} \tilde\sigma^2(x_k, \bar x_j) \prod_{j=1}^{K_{\bar 1}} \frac{1 - \frac{1}{x_k^- y_{\bar 1,j}}}{1 - \frac{1}{x_k^+ y_{\bar 1,j}}} \prod_{j=1}^{K_{\bar 3}} \frac{1 - \frac{1}{x_k^- y_{\bar 3,j}}}{1 - \frac{1}{x_k^+ y_{\bar 3,j}}},\end{aligned} \tag{4.17}$$

$$1 = \prod_{j=1}^{K_2} \frac{y_{3,k} - x_j^+}{y_{3,k} - x_j^-} \prod_{j=1}^{K_{\bar 2}} \frac{1 - \frac{1}{y_{3,k}\bar x_j^-}}{1 - \frac{1}{y_{3,k}\bar x_j^+}}, \tag{4.18}$$

$$1 = \prod_{j=1}^{K_{\bar 2}} \frac{y_{\bar 1,k} - \bar x_j^-}{y_{\bar 1,k} - \bar x_j^+} \prod_{j=1}^{K_2} \frac{1 - \frac{1}{y_{\bar 1,k} x_j^+}}{1 - \frac{1}{y_{\bar 1,k} x_j^-}}, \tag{4.19}$$

$$\begin{aligned}\left(\frac{\bar x_k^+}{\bar x_k^-}\right)^L = &\prod_{\substack{j=1\\ j\neq k}}^{K_{\bar 2}} \frac{\bar x_k^- - \bar x_j^+}{\bar x_k^+ - \bar x_j^-} \frac{1 - \frac{1}{\bar x_k^+ \bar x_j^-}}{1 - \frac{1}{\bar x_k^- \bar x_j^+}} \sigma^2(\bar x_k, \bar x_j) \prod_{j=1}^{K_{\bar 1}} \frac{\bar x_k^+ - y_{\bar 1,j}}{\bar x_k^- - y_{\bar 1,j}} \prod_{j=1}^{K_{\bar 3}} \frac{\bar x_k^+ - y_{\bar 3,j}}{\bar x_k^- - y_{\bar 3,j}} \\ &\times \prod_{j=1}^{K_2} \frac{1 - \frac{1}{\bar x_k^- x_j^-}}{1 - \frac{1}{\bar x_k^+ x_j^+}} \frac{1 - \frac{1}{\bar x_k^+ x_j^-}}{1 - \frac{1}{\bar x_k^- x_j^+}} \tilde\sigma^2(\bar x_k, x_j) \prod_{j=1}^{K_1} \frac{1 - \frac{1}{\bar x_k^+ y_{1,j}}}{1 - \frac{1}{\bar x_k^- y_{1,j}}} \prod_{j=1}^{K_3} \frac{1 - \frac{1}{\bar x_k^+ y_{3,j}}}{1 - \frac{1}{\bar x_k^- y_{3,j}}},\end{aligned} \tag{4.20}$$

$$1 = \prod_{j=1}^{K_{\bar 2}} \frac{y_{\bar 3,k} - \bar x_j^-}{y_{\bar 3,k} - \bar x_j^+} \prod_{j=1}^{K_2} \frac{1 - \frac{1}{y_{\bar 3,k} x_j^+}}{1 - \frac{1}{y_{\bar 3,k} x_j^-}}. \tag{4.21}$$

What are these equations

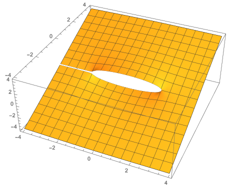
- A simplified example of the equations:

$$\eta = \pm 1, 0$$

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{\substack{j=1\\ j\neq k}} \left(\frac{x_k^+ - x_j^-}{x_k^- - x_j^+}\right)^{\eta} \frac{1-\frac{1}{x_k^+ x_j^-}}{1-\frac{1}{x_k^- x_j^+}} (\sigma^{\bullet\bullet})^2(x_k, x_j)$$

- x is the so-called Zhukovsky variable

$$x + \frac{1}{x} = \frac{u}{g}$$



$x(u) = \frac{1}{2g}(u + \sqrt{u+2g}\sqrt{u-2g})$

- $\sigma^{\bullet\bullet}(x, y)$ is the dressing phase

Large Volume Checks

- For large volume approximation of QSC we need 2 assumptions:
  - Q-functions scale as their asymptotics
  - $\mu_1{}^{\dot{2}}, \mu_{\dot{1}}{}^{2}$ have square root cuts!
- In the limit it is possible to find a subset of Q-functions explicitly. Example:

$$\mathbf{P}_1 \propto x^{-\frac{L}{2}} \Sigma$$

- ABA is then reproduced from QQ-relations and the additional dressing phases are constrained by $\mathbf{P}_a^{\gamma} = \mathbf{P}_{\dot{a}} \mu^{\dot{a}}{}_a$ to satisfy crossing equations.
- The crossing equations match exactly those found in the study of the AdS_3/CFT_2 S-matrix [Borsato,Ohlsson Sax,Sfondrini,Stefanski,Torielli '13] (see also [Frolov,Sfondrini '21])

Summarizing AdS_3/CFT_2 QSC

- I have presented a proposal for a Quantum Spectral Curve for $AdS_3\times S^3\times T^4$ based on symmetry considerations and lessons from previous curves. (Procedure presented Monodromy bootstrap. [Cavaglià,Gromov,Stefanski,Torrielli, 21'] found it independently.)
- The curve reproduces the ABA in the large volume limit and gives the correct crossing equations.
- So far I have only discussed massive modes. We expect massless to also be included by slightly changing some analytic properties.
- Extent of validity of ABA should be in question. Important to actually solve the curve.

Solving the AdS$_3$ QSC

What can we calculate

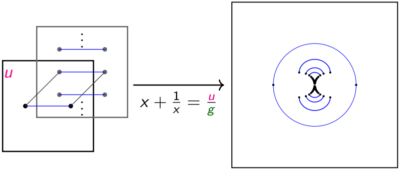
- What are the challenges?
  - No results to compare against: Need both numerics and analytic weak coupling to double check.
  - All analytical methods used in the past relies on quadratic branch cuts.

Review of $\mathcal{N}=4$ recipe

- First: Quick review of numerical and analytical methods in $\mathcal{N}=4$ [Marboe,Volin'14, Gromov,Levkovich-Maslyuk,Sizov '15].
- Step 1: Ansatz for $\mathbf{P}_a$:

Parameters to fix!

$$\mathbf{P}_a \propto \sum_{n=-M_a}^{\infty} \boxed{\frac{c_{a,n}}{x^n}}$$



- Step 2: Reconstruct μ_{ab}:

$$\text{RH Problem: } \mu_{ab}^{\gamma}-\mu_{ab}=\mathbf{P}_a\mathbf{P}_b^{\gamma}-\mathbf{P}_a^{\gamma}\mathbf{P}_b$$

$$\text{FD Equation: } \mu_{ab}^{[2]}=(\delta_a^c-\mathbf{P}_a\mathbf{P}^c)\mu_{cd}(\delta_b^d+\mathbf{P}^d\mathbf{P}_b)$$

- Step 3: Impose $\mathbf{P}\mu$

$$\mathbf{P}_a^{\gamma}=\mu_{ab}\mathbf{P}^b \qquad \mathbf{P}^a=\underbrace{\chi^{ab}}_{\begin{pmatrix}0&0&0&-1\\0&0&1&0\\0&-1&0&0\\1&0&0&0\end{pmatrix}}\mathbf{P}_a$$

AdS$_3$

- Now to AdS$_3$. Simple "$\mathfrak{sl}_2$ sector". ($\mathcal{N}=4$ analogue $\mathrm{tr}\big(Z\nabla^{\mathcal{S}}Z\big)+\ldots$,$\mathcal{S}$ even).
- Technical challenge: $\mathbf{P}_a, \mathbf{P}_{\dot{a}}$ does not have square-root cuts

$$\mathbf{P}_a \propto \sum_{n=-M_a}^{\infty} \frac{c_{a,n}}{x^n}$$

Not good near cut

- Can we find another object? Consider

$$\mathbf{P}_a^{2\gamma} = W_a{}^b \mathbf{P}_b \,, \tag{5.1}$$

where

Regularised μ

$$(\mu^R)_a{}^{\dot{b}} = \mu_a{}^{\dot{b}} + \mathbf{P}_a (\mathbf{P}^{\dot{b}})^{\bar{\gamma}}$$

$$W_a{}^b = (\mu^R)_a{}^{\dot{c}} (\mu^R)_{\dot{c}}{}^b .$$

- We can restore $\mu_a{}^b$ given the gluing matrix $N_k{}^l$. It is defined by

$$\mathbf{Q}_k^{\gamma}(u) = N_k{}^l \mathbf{Q}_l(-u) \,. \tag{5.2}$$

We considered only cases with $N_k{}^l$ off-diagonal.

Formulating a weak coupling algorithm

- Introduce a new object

$$\mathbb{P}_a = (W^{\ell})_a{}^b \mathbf{P}_b \,, \qquad \ell^{2\gamma} = \ell - 1 \,. \tag{5.3}$$

- Then $\mathbb{P}_a$ is a quadratic cut function. It can be parameterised as

$$\mathbb{P}_a = \sum_{k=-\infty}^{\infty} \frac{d_{a,k}}{x^k} \,.$$

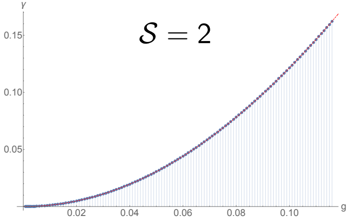
- Trick: Compute $\mathbf{P}_a^{\gamma}$ in two different ways.

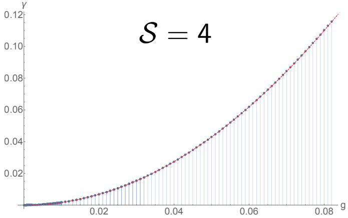
$$\mathbb{P}_a^{\gamma} = \sum_{k=-\infty}^{\infty} d_{a,k} x^k \,, \quad \mathbb{P}_a^{\gamma} = (W^{\ell^{\gamma}})_a{}^b (\mu^R)_b{}^{\dot{c}} \mathbf{P}_{\dot{c}} = \sum_{k=-\infty}^{\infty} \frac{\tilde{d}_{a,k}}{x^k} \,.$$

Find point where $d_{a,k} = \tilde{d}_{a,-k}$ using numerics.

Numerical results for $\mathcal{S} = 2, 4$

- Implementing the numerical algorithm





- From numerical data we find the following ansatz:

$$\mathbf{P}_a \simeq_{x\to\infty} \sum_{a=-M_a}^{\infty} \frac{c_{a,n}}{x^n}$$

$$\mathbf{P}_a \simeq_{x\to 1} \sum_n \frac{d_{a,n}^{(0)}}{x^n} + g \sum_n \frac{d_{a,n}^{(1)}}{x^n} \log\left(\frac{x-1}{x+1}\right) + g^2 \sum_n \frac{d_{a,n}^{(2)}}{x^n} \log^2(\frac{x-1}{x+1}) + \ldots$$

Analytic Solution

- Inverting W allows us to write

$$\mathbf{P}_a = \sum_{n=0}^{\infty} \frac{(-\ell)^n}{n!} \log(W)_a{}^b \mathbb{P}_b \,, \tag{5.4}$$

- With this ansatz one proceeds to calculate μ and imposes the $\mathbf{P}\mu$-system.
- This closes the system and we find the anomalous dimension!

Analytic $\mathcal{S} = 2$ results

- We found the following result

$$\gamma_{\mathcal{S}=2} = 12\, g^2 + \frac{864}{35\pi} g^3 + (-48 - \frac{576}{7\pi^2}) g^4 + \left(-\frac{405504}{875\pi^3} - \frac{51552}{143\pi}\right) g^5$$
$$+ (444 - \frac{70665216}{4375\pi^4} + \frac{230121984}{175175\pi^2}) g^6$$
$$+ (-\frac{16896}{35\pi}\zeta_3 - \frac{4965482496}{21875\pi^5} + \frac{6791453184}{875875\pi^3} + \frac{1102677696}{146965\pi}) g^7$$
$$+ (-288\zeta_3 + \frac{1898496}{1225\pi^2}\zeta_3 - 576\zeta_5 - 5844$$
$$- \frac{302725824512}{109375\pi^6} - \frac{9030729728}{25025\pi^4} + \frac{25695082110528}{282907625\pi^2}) g^8$$

- Comparing with ABA

$$\gamma^{ABA}_{\mathcal{S}=2} = 12 g^2 + \mathcal{O}(g)^4 \implies \frac{864}{35\pi} g^3 \quad \text{massless wrapping?}$$

- Terms in brown agrees with $\mathcal{N} = 4$, $\gamma_{\mathcal{S}=4} \rightarrow_{\pi\rightarrow\infty} = \gamma^{\mathcal{N}=4}_{\mathcal{S}=2} + \mathcal{O}(g)^6$

More general $\mathcal{S}$ results

- Using the same procedure we can find $\gamma_{\mathcal{S}}$ for $\mathcal{S}$ even.
- Some easy patterns? Yes! Write

  $$\gamma_{\mathcal{S}} = f_{(2)}(\mathcal{S})g^2 + f_{(3)}(\mathcal{S})g^3 + f_{(4)}(\mathcal{S})g^4 + \mathcal{O}(g^5)\,,$$

  then from data for $S = 2, 4, 6, 8$ we found the following results

  $$f_{(2)}(\mathcal{S}) = 8\,S_1(\mathcal{S}) \qquad f_{(3)}(\mathcal{S}) = \tfrac{384}{35\pi}\,S_1(\mathcal{S})^2$$

  as well as

  $$f_{(4)}(\mathcal{S}) = \Delta_4^{\mathcal{N}=4} - \frac{512}{21\pi^2}S_1(S)^3\,. \qquad (5.5)$$

  with $S_1(S) = \sum_{k=1}^{S} \frac{1}{k}$.
- Curiously the the expansion in g does not seem to commute with $S \to \infty$.

Summarizing Solving the Curve

- While the proposed curve remains challenging it can be solved.
- Using a variety of techniques $\gamma_{\mathcal{S}=2}$ has been calculated up to g^8. The result contains ζ-values and inverse powers of π.
- The method used extends to even $\mathcal{S}$. Will be important to also eventually study $\mathfrak{su}_2$ sectors and more general settings.
- Note that while the QSC is a conjecture it is a very sharp conjecture.
- Many open questions:
    - Can we find "massless wrapping terms" from standard Luscher corrections?
    - Strong coupling numerically?
    - Can we be sure there are no excited massless states?

Conclusions and Outlook

Conclusions

- The QSC for $\mathcal{N} = 4$ is a powerful-integrability tool, it should be extended beyond $\mathcal{N} = 4$,
- Using the structures of Q-systems it is possible to construct new QSC's. A method to do so is Monodromy Bootstrap.
- Coupling two $\mathfrak{psu}(1,1|2)$ Q-systems using Monodromy Bootsrap gives a curve which conjecturally describes the spectrum, in the planar limit, of string theory on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$.
- While challenging the conjectured curved can be solved both numerically and analytically in the weak coupling regime. This gives very precise predictions for the anomalous dimension.

Outlook

- It would be interesting to see if the single $\mathfrak{psu}(1,1|2)$ spectral could describe AdS$_2$/CFT$_1$ [Sorokin,Tseytlin,Wulff,Zarembo '11]. Can we use results from [Hoare, Pittelli, Torrielli '14,'15]?
- It would be very interesting to study QSCs based on symmetries beyond $\mathfrak{gl}_{m|n}$. In particular, $D_{2,1|\alpha}$ is relevant for string theory on AdS$_3$×S^3×S^3×S^1.
- It would be very interesting to compare our precise predictions with the AdS$_3$ TBA [Frolov,Sfondrini '21].
- Massless modes still need clarification, would be very interesting to compare with [Brollo,le Plat,Sfondrini,Suzuki '23]
- We could try to deform the curve in various ways:
  - Fishchain? [Gromov,Sever,'19]
  - Deformations towards NS-flux sector [Hoare,Tseytlin '13]. (Compare with [Eberhardt,Gaberdiel,Gopakumar '18,'19])

Thank You

Thank You!