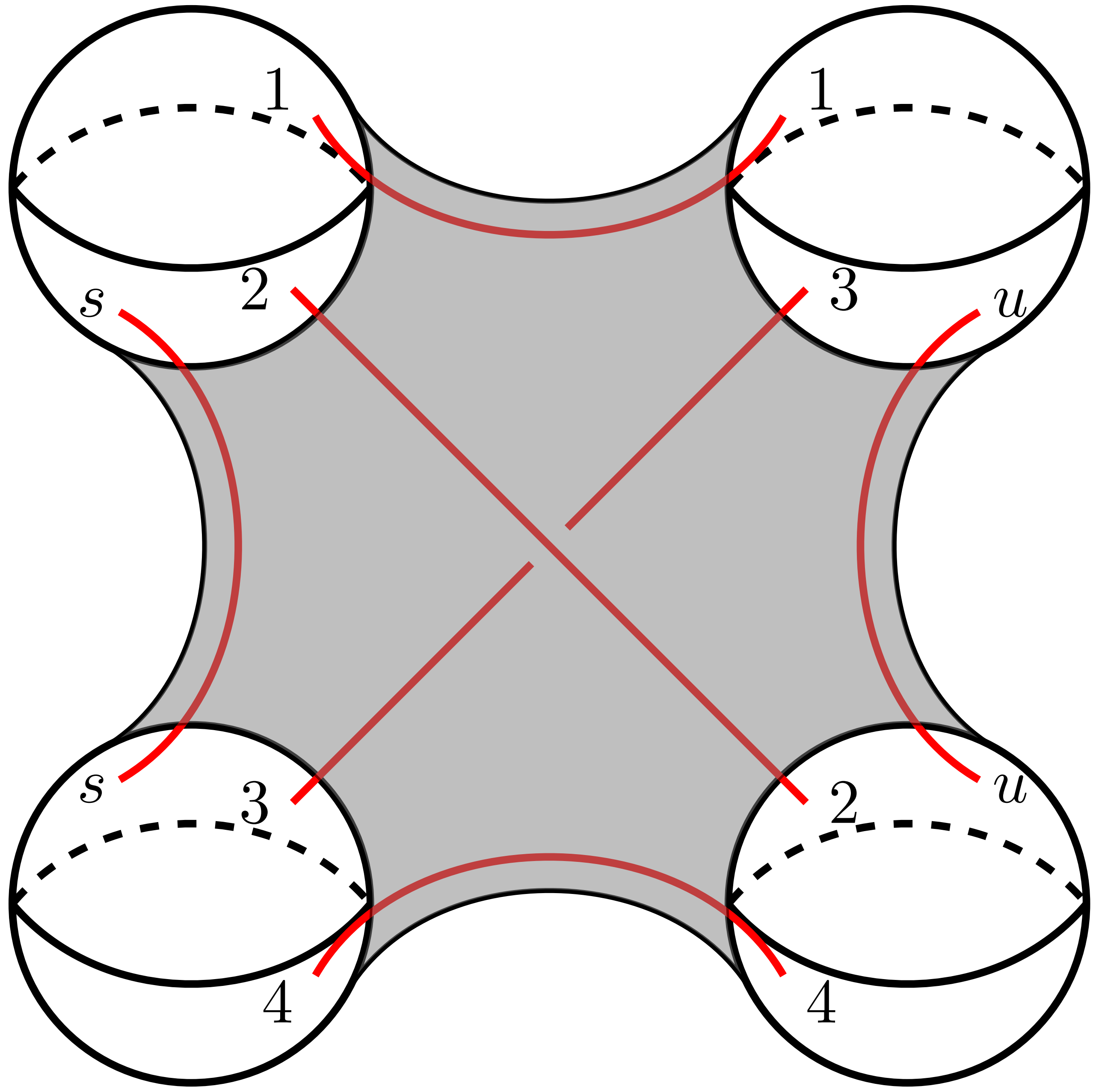
3d quantum gravity & Virasoro TQFT



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2024/05/14

mostly based on **2304.13650** & **2401.13900** with Lorenz Eberhardt and Mengyang Zhang

The gravitational path integral

- A central goal in theoretical physics is to formulate gravity quantum mechanically (black hole information problem, early universe cosmology, …)
- Many recent approaches have centered on the **Euclidean gravitational path integral**:

$$Z_{\text{grav}}(\partial M) = \sum_{\text{topologies M}} \int \frac{[dg]}{\text{Diff(M)}} e^{-S_{\text{grav}}[g]}$$

- In most situations, the rules governing this computation are unclear (What topologies/geometries to include (singularities)? When is the saddle-point approximation valid? How to go beyond the semiclassical saddle-point expansion? Off-shell configurations? …)
- In practice most computations done in semiclassical gravity
- On the other hand, as typically formulated worldsheet string theory is not well-suited to these sorts of non-perturbative questions

“Pure” quantum gravity?

- One might hope that it would be possible to retreat from string theory
  - Is it possible to isolate the essential physics and consistently formulate a simpler theory: **“pure quantum gravity”**?
  - The dual boundary system would describe the physics of gravitons and black hole microstates holographically
  - If one could make this precise, it would provide a clean theoretical laboratory for fundamental questions in quantum gravity and quantum aspects of black holes
- Designed to probe the question of what are the minimal set of ingredients needed for a consistent formulation of quantum gravity

Recent advances in low-dimensional quantum gravity

- “Nearly pure” theories of two-dimensional dilaton quantum gravity have been the focus of a significant body of work in recent years
- The most famous example is **JT gravity** (and deformations thereof) [Jackiw; Teitelboim; Almheiri, Polchinski; Jensen; Maldacena Stanford Yang; Engelsoy Mertens Verlinde; …]

$$S_{\rm JT}[\Sigma;\Phi,g] = -\underbrace{\frac{1}{2}\int_\Sigma d^2x\sqrt{g}\,\Phi(R+2)}_{\text{sets } R=-2} - \underbrace{\int_{\partial\Sigma} dx\sqrt{h}\,\Phi(K-1)}_{\text{boundary action}} - \underbrace{S_0\,\chi(\Sigma)}_{\text{weights topology}}$$

- Arise universally in the near-extremal limit of higher-dimensional black holes
- Holographic dual is a double-scaled **random matrix integral**; a statistical ensemble of boundary Hamiltonians rather than a particular quantum system (!) [Saad Shenker Stanford 2019]

AdS$_3$ Einstein gravity

- Perhaps the simplest theory of gravity that one might expect to have a local boundary dual is three-dimensional Einstein gravity with negative cosmological constant (possibly coupled to massive point particles)

$$S_{\mathrm{grav}}(M; g) = -\frac{1}{16\pi G_N}\int_M d^3x\sqrt{g}\left(R + \frac{2}{\ell^2}\right) + (\text{boundary term})$$

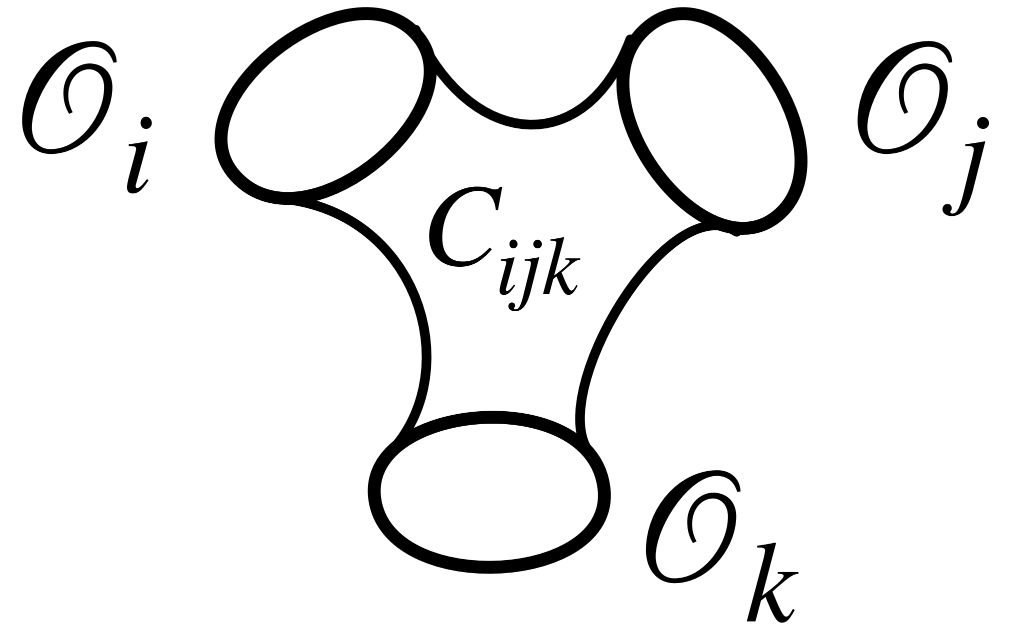
- **Naively trivial**
  - all solutions locally identical: $\mathbb{H}^3/\Gamma$
  - no propagating gravitational degrees of freedom (no gravitational waves)
- ... **But it's not!**
  - Boundary gravitational dynamics ("gravitational edge modes")
  - Infinite-dimensional asymptotic conformal symmetry: boundary 2d CFT with $c = 3\ell/2G_N$ [Brown Henneaux 1986]
  - Crucially, there exist non-trivial **black hole solutions** [Banados Teitelboim Zanelli 1992]

What would it mean to solve it?

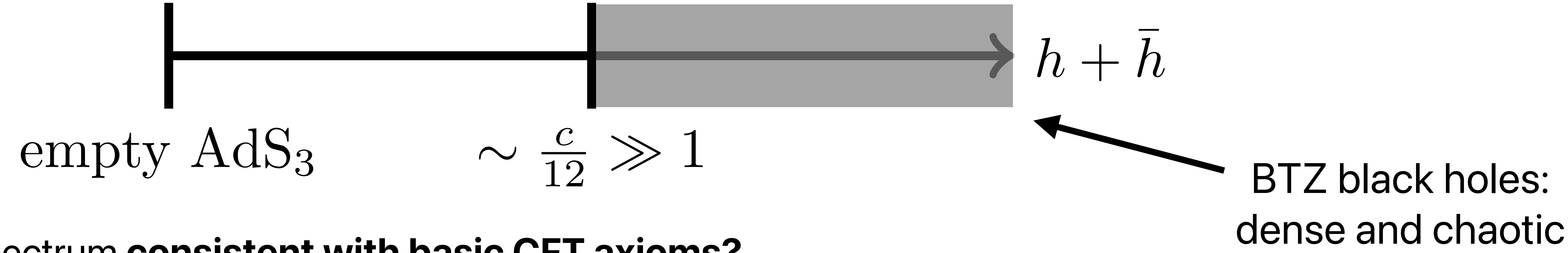
What would it mean to solve it?

- Conventional wisdom:
  Construct the dual 2D conformal field theory (ideally a large-c family of such theories)

- CFT solved in terms of **CFT data**:



  - spectrum of local (primary) operators $\{\mathcal{O}_{h,\bar{h}}\}$
  - their dynamics $\left\{\langle \mathcal{O}_i \mathcal{O}_j \mathcal{O}_k \rangle = C_{ijk}\right\}$, "structure constants"



- Is this kind of spectrum **consistent with basic CFT axioms?**
  [Hellerman 2009; SC Lin Yin 2016; Afkhami-Jeddi Hartman Tajdini 2019; Hartman Mazac Rastelli 2019; …]

- Despite significant work that has been done to carve out the space of possibilities, nothing remotely close to this kind of CFT has been constructed (but not ruled out either)

Input from the low-energy effective theory

- Black holes obey the laws of thermodynamics and are characterized by the Bekenstein-Hawking entropy

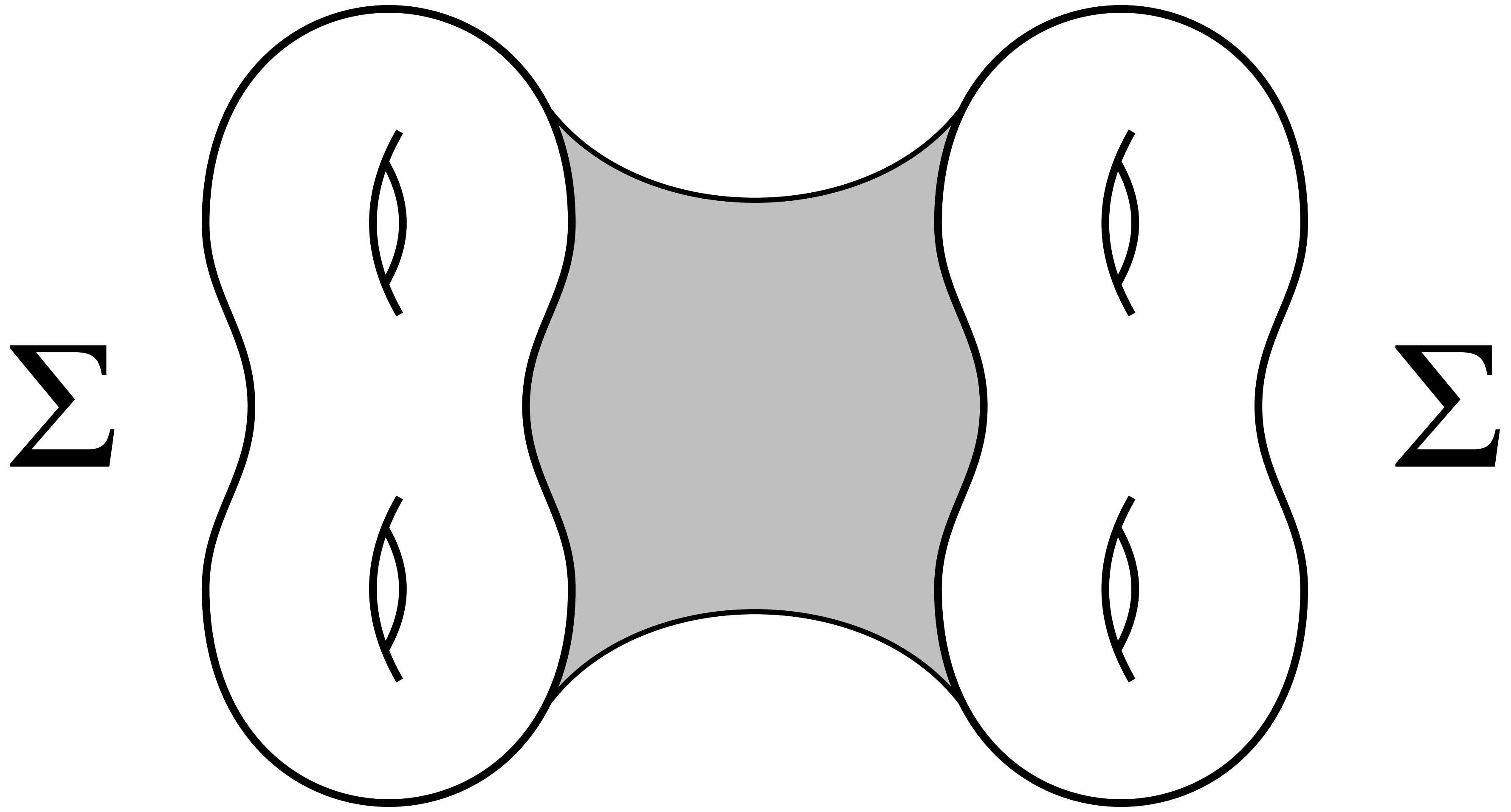
$$S_{\rm BH} = \frac{A}{4G_N}$$

- This entropy may be computed in the low-energy effective theory from the contribution to the gravitational path integral from the Euclidean black hole saddle
  - A hint from the low-energy effective theory about coarse-grained density of states of the microscopic UV completion, e.g. [Strominger Vafa 1996; Strominger 1997]

$$e^{S_{\rm BH}} \leftrightarrow \overline{\rho(h,\bar{h})}$$

Euclidean wormholes

- AdS_3 gravity also admits multi-boundary **Euclidean wormholes** as bona fide solutions to Einstein's equations [Maldacena Maoz 2004]



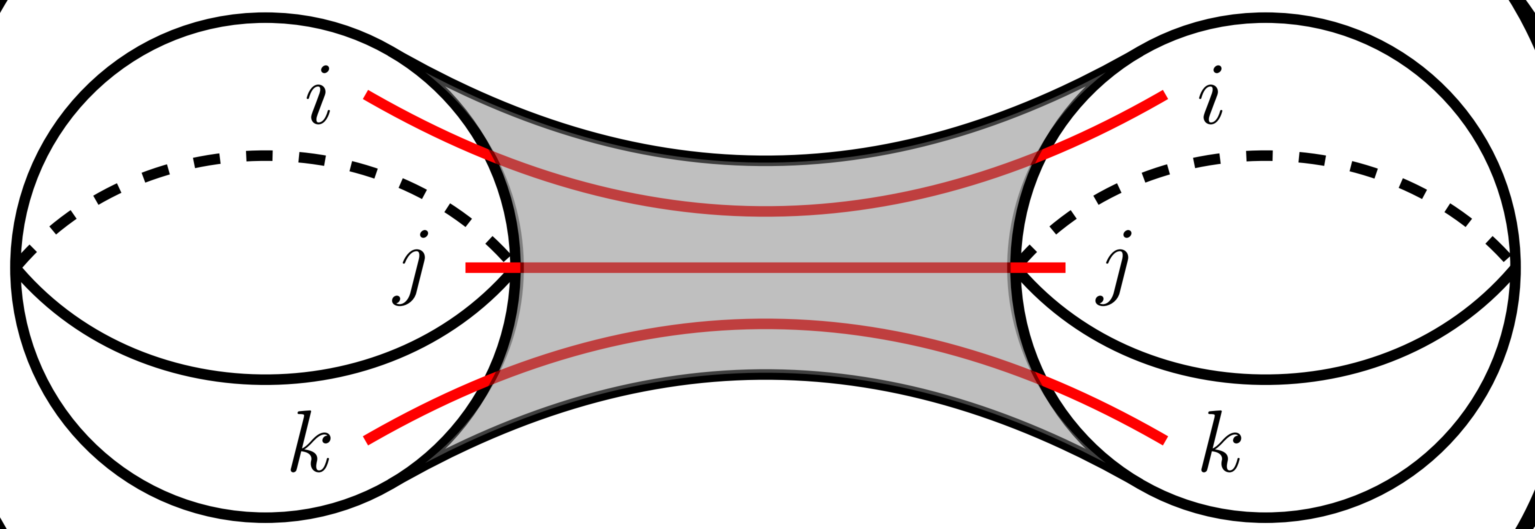
$$ds^2 = d\rho^2 + \cosh^2\rho \, d\Sigma^2$$

- These are conceptually puzzling for the traditional holographic paradigm: non-factorizing contributions to Z_{grav} are naively inconsistent with specific boundary quantum system
  - "factorization problem"

"Ensemble of CFT data"

- Euclidean wormholes provide a **low-energy window into the space of UV completions** in the same sense as the Euclidean black hole saddle & Bekenstein-Hawking entropy: **encode coarse-grained statistical properties of CFT data** [Chandra SC Hartman Maloney 2022]

e.g. $e^{-S_{\text{grav}}}\Big($



$\Big) \equiv \overline{C_{ijk}^2}$

Universal Asymptotic formula $|C_0(P_i, P_j, P_k)|^2$ [SC Maloney Maxfield Tsiares 2019]

- Pseudorandom nature of high energy states of chaotic systems motivates the following random ansatz for CFT data ("Virasoro ETH"):

$$C_{ijk} = \sqrt{\overline{C_{ijk}^2}}\, R_{ijk}$$

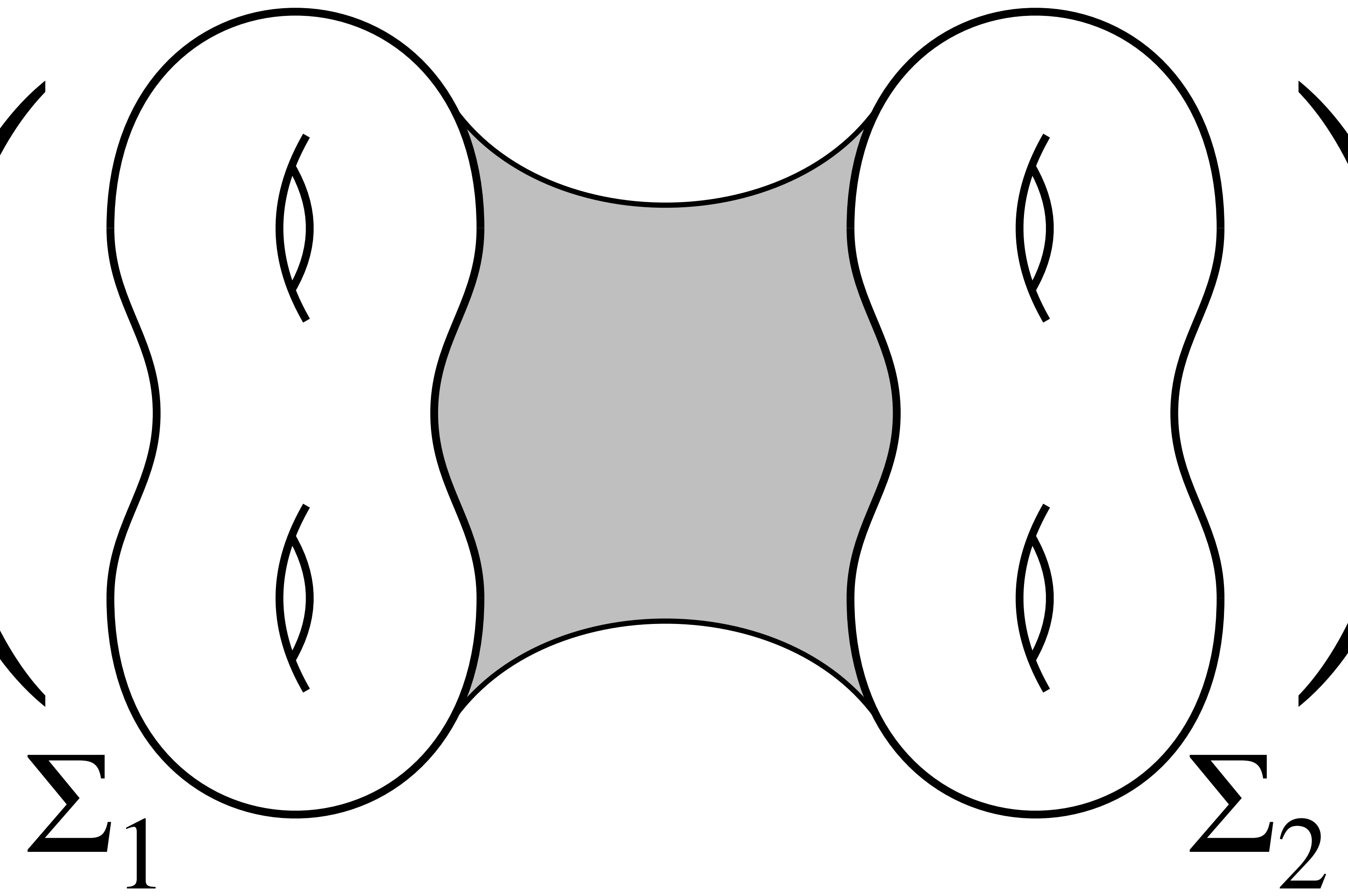
Random tensor with Gaussian statistics

- More provocatively, one might interpret Euclidean wormholes as computing averaged CFT quantities in a putative ensemble holographic dual

“Ensemble of CFT data”

- Indeed, averaged CFT quantities in this Gaussian ensemble of CFT data reproduce saddle-point actions (+ one-loop effects) in semiclassical AdS_3 gravity [Chandra SC Hartman Maloney 2022]

$$e^{-S_{\text{grav}}}\left(\quad \right) \approx \overline{Z_{\text{CFT}}(\Sigma_1) Z_{\text{CFT}}(\Sigma_2)}$$



- But this cannot be the full story
  - Important difference from JT/RMT duality:
    Boundary theory is subject to an infinite set of non-perturbative consistency conditions (locality), of which there are large violations in the Gaussian ensemble [Belin de Boer Jafferis Nayak Sonner 2023]
  - Not even clear in principle how an “ensemble of CFTs” or “random CFT” should be defined;
    huge gap in our knowledge of irrational 2d CFTs!

Can we make sense of the bulk theory in its own right?

Today

- Despite recent progress the status of the holographic dual of pure AdS_3 gravity is **unresolved**
- Although not trivial, the bulk behaves in many ways like a topological theory
  - Topological quantum field theory (TQFT) hence provides a natural framework for the exact formulation of the bulk theory
- Today I will discuss the canonical quantization of the theory and will concretely relate it to a novel TQFT: **“Virasoro TQFT”**
- Explicit and useful!
  - Completely solves the gravity theory at the level of fixed (on-shell) topologies (hyperbolic 3-manifolds)
  - But puzzles having to do with the sum over topologies and the contributions of off-shell configurations still largely remain

Plan

- The gravity phase space
- Quantization of the phase space and “Virasoro TQFT”
- Computing 3d gravity partition functions: general procedure and examples

The gravity phase space

- AdS_3 gravity famously admits a classical reformulation in terms of a topological gauge theory: (two copies of) $SL(2,\mathbb{R})$ **Chern-Simons theory** [Achucarro Townsend 1986; Witten 1988, …]

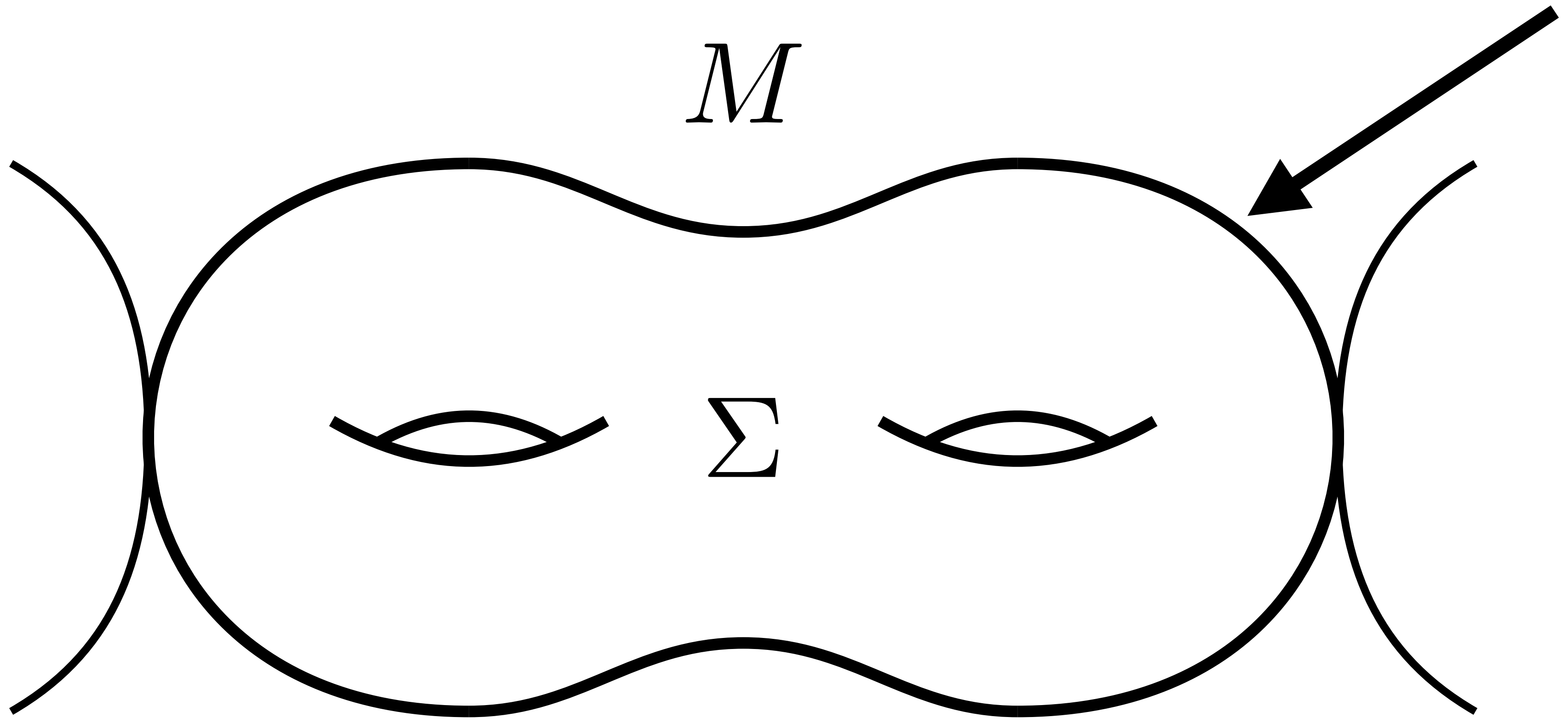
$$\mathscr{A}^a_\mu = \omega^a_\mu + \frac{1}{\ell} e^a_\mu, \quad \overline{\mathscr{A}}^a_\mu = \omega^a_\mu - \frac{1}{\ell} e^a_\mu$$

$SL(2,\mathbb{R})$ gauge fields — Dualized spin connection — Vielbein

- Einstein equations $\rightarrow \mathscr{A}, \overline{\mathscr{A}}$ are **flat connections**
- But the phase spaces of the two theories are importantly different!
  - More conceptual issues:
    large diffeos gauged in gravity, sum over topologies in gravity

The gravity phase space

- Consider the canonical quantization of 3d gravity on a spatial surface Σ



- Gauge theory phase space: $\mathscr{M}_{\rm CS}(\Sigma) =$ moduli space of flat connections on Σ
  - Most such connections are not good initial conditions for gravity!
- Gravity phase space: $\mathscr{M}_{\rm grav}(\Sigma) = \underbrace{\mathscr{T}(\Sigma)} \times \overline{\mathscr{T}(\Sigma)} \subset \mathscr{M}_{\rm CS}(\Sigma)$

"Teichmuller space": space of hyperbolic metrics on Σ

(uplift to smooth 3d metrics on M)

[Krasnov Schlenker 2005; Scarinci Krasnov 2011; Kim Porrati 2015]

The Hilbert space and quantum Teichmuller theory

- In order to proceed to the quantum theory, our task is now to **quantize Teichmuller space**
- This was carried out in [H. Verlinde 1989] (later refined by [Kashaev 1998, Teschner 2002-2005]):

$$\mathscr{H}_{\text{grav}}(\Sigma) = \mathscr{H}(\Sigma) \times \overline{\mathscr{H}(\Sigma)}$$

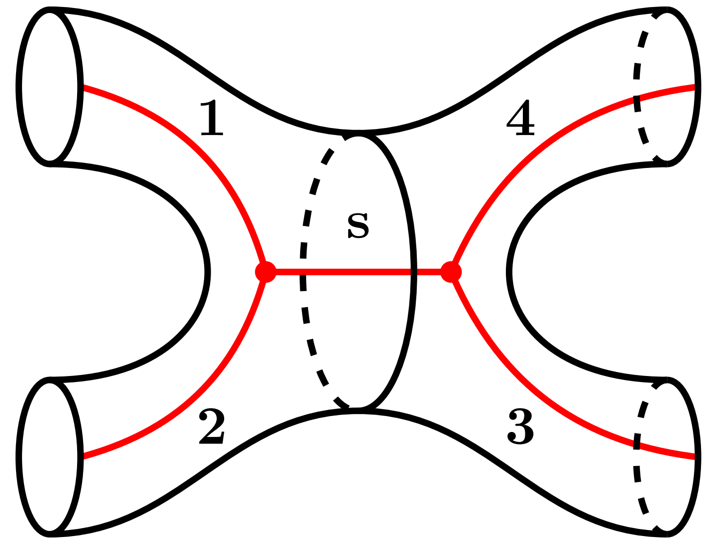
where $\mathscr{H}(\Sigma) = \left\{ \textbf{Virasoro conformal blocks} \text{ on } \Sigma \text{ with } h_i \geq \frac{c-1}{24} \right\}$

- Conformal blocks are a natural family of wavefunctions on Teichmuller space
- Intuitive from the point of view of holography:
  conformal blocks are the **building blocks of CFT correlation functions** on Σ
- The Hilbert space is infinite-dimensional
- These are the same conformal blocks that appear in **Liouville CFT**

Conformal blocks

- Conformal blocks are the kinematic building blocks of local CFT observables
- Algorithm: pair of pants decomposition of $\Sigma \to$ insertion of complete sets of states on cuffs $\to$ structure constants for each pair of pants

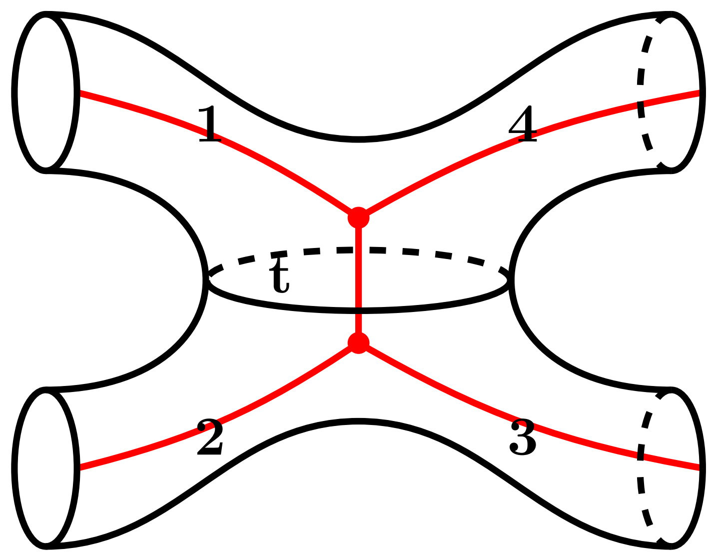
$$\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle_{g=0} = \sum_{\mathcal{O}_s} \underbrace{C_{12s}C_{s34}}_{\text{CFT data}} \underbrace{\Big| \quad \Big|^2}_{\text{conformal blocks } \mathcal{F}_{\Sigma_{0,4}}(P_s\,|\,m)}$$



$$h_s = \frac{c-1}{24} + P_s^2$$

conformal blocks $\mathcal{F}_{\Sigma_{0,4}}(P_s\,|\,m)$

$$= \sum_{\mathcal{O}_t} C_{14t}C_{23t} \Big| \quad \Big|^2$$



- Independence of pants decomposition: infinite set of non-perturbative constraints on CFT data

The inner product in Virasoro TQFT

- An explicit form of the inner product on Teichmuller space based on geometric quantization was proposed in [H. Verlinde 1989].
  A modern reinterpretation [SC Eberhardt Zhang 2023]:

Internal conformal weights: $h_i = \frac{c-1}{24} + P_i^2$

Moduli

$$\langle \mathcal{F}_\Sigma(\mathbf{P}) \,|\, \mathcal{F}_\Sigma(\mathbf{P}') \rangle = \int_{\mathcal{T}(\Sigma)} d\mathbf{m} \left(Z_{\text{bc ghost}} \, Z_{\text{TL}}^{\hat{c}=26-c} \right) \overline{\mathcal{F}_\Sigma(\mathbf{P}\,|\,\mathbf{m})} \, \mathcal{F}_\Sigma(\mathbf{P}'\,|\,\mathbf{m})$$

Conformal blocks on Σ

b, c ghost partition function/correlator

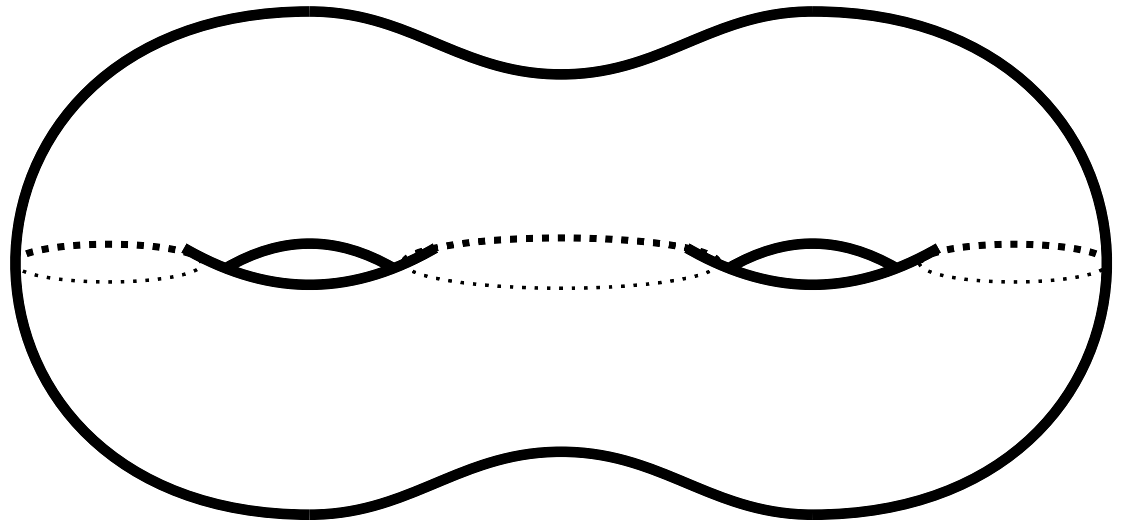
Partition function/correlator in "timelike Liouville CFT"

- Structure is reminiscent of a perturbative string amplitude in a funny **worldsheet string theory**
  - Integrating individual conformal blocks (not crossing invariant) over Teichmuller space rather than full partition functions/correlators over moduli space
  - Indeed, **"Virasoro minimal string theory"** arises via dimensional reduction [SC, Eberhardt, Mühlmann, Rodriguez 2023]

The inner product in Virasoro TQFT

- A main contribution of our paper was to give a more useful representation of this inner product:

$$\langle \mathscr{F}_\Sigma(\mathbf{P}) \,|\, \mathscr{F}_\Sigma(\mathbf{P}') \rangle = \frac{\delta^{3g-3+n}(\mathbf{P} - \mathbf{P}')}{\rho_\Sigma(\mathbf{P})}$$



where ρ_Σ is the **OPE density in Liouville CFT** on Σ

$$\rho_\Sigma(\mathbf{P}) = \left(\prod_{a=1}^{3g-3+n} \rho_0(P_a) \right) \left(\prod_{\text{pairs of pants (i,j,k)}} C_0(P_i, P_j, P_k) \right)$$

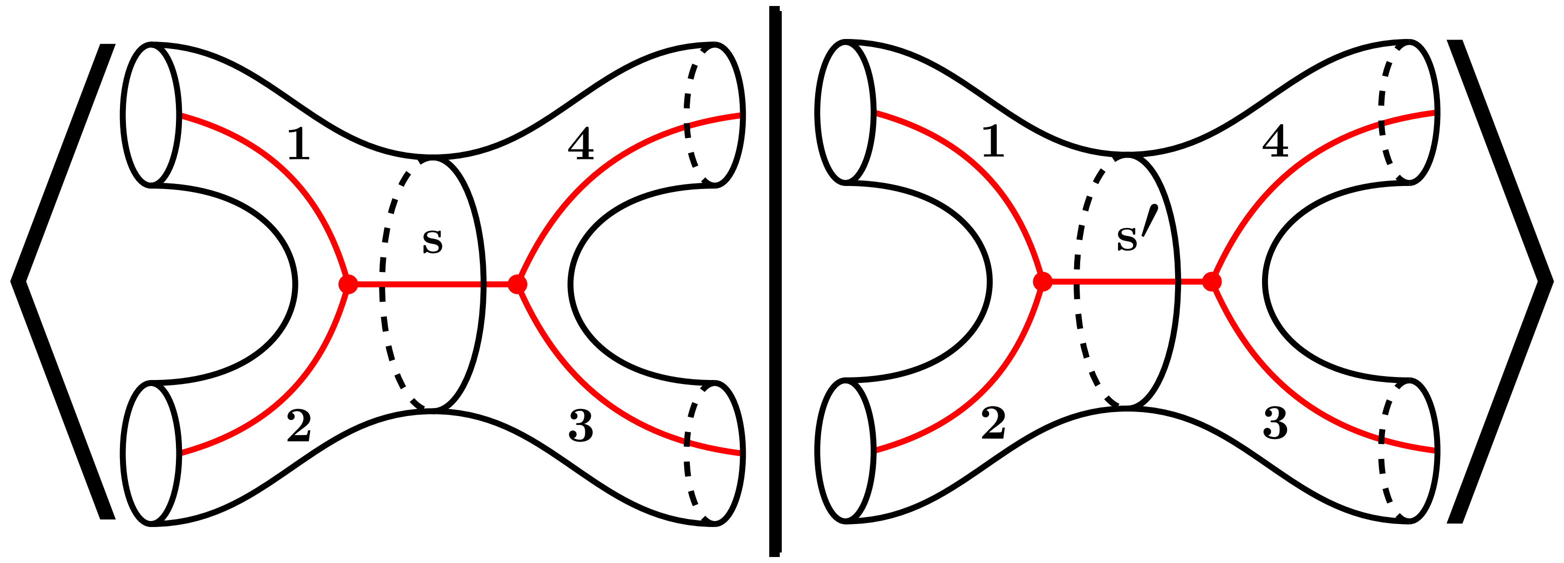
Internal cuffs

DOZZ formula for Liouville structure constants

- Can be argued for from various perspectives: consistency with crossing; bootstrap; 3d gravity
- Note: the identity block (and all blocks involving the exchange of sub-threshold operators) are **non-normalizable states** in the Hilbert space $\mathscr{H}_\Sigma$

For example

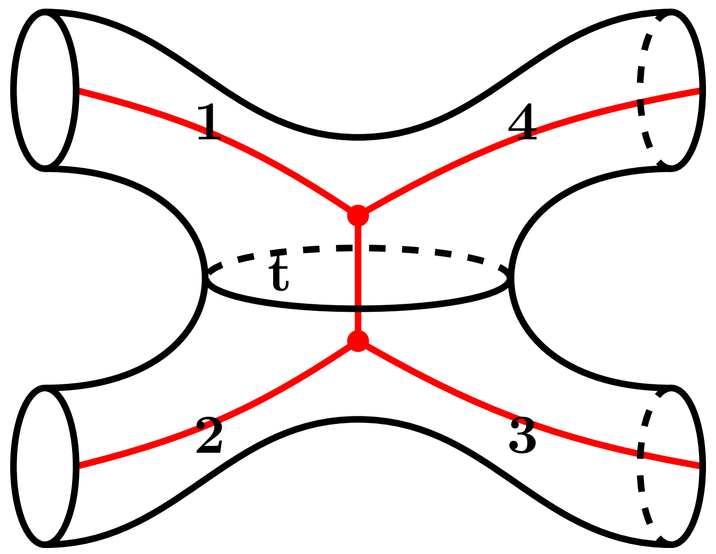
- For concreteness, consider the inner product between sphere four-point Virasoro blocks:



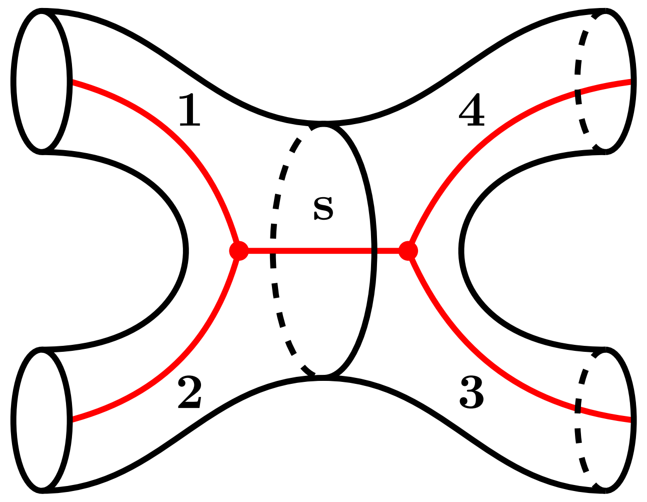
$$= \frac{\delta(P_s - P_s')}{\rho_0(P_s)C_0(P_1, P_2, P_s)C_0(P_3, P_4, P_s)}$$

Crossing

- Conformal blocks transform nicely under change of pants decomposition ("crossing transformations")



$$= \int_0^\infty dP_s \, \mathbb{F}_{P_t P_s} \begin{bmatrix} P_1 & P_2 \\ P_4 & P_3 \end{bmatrix}$$

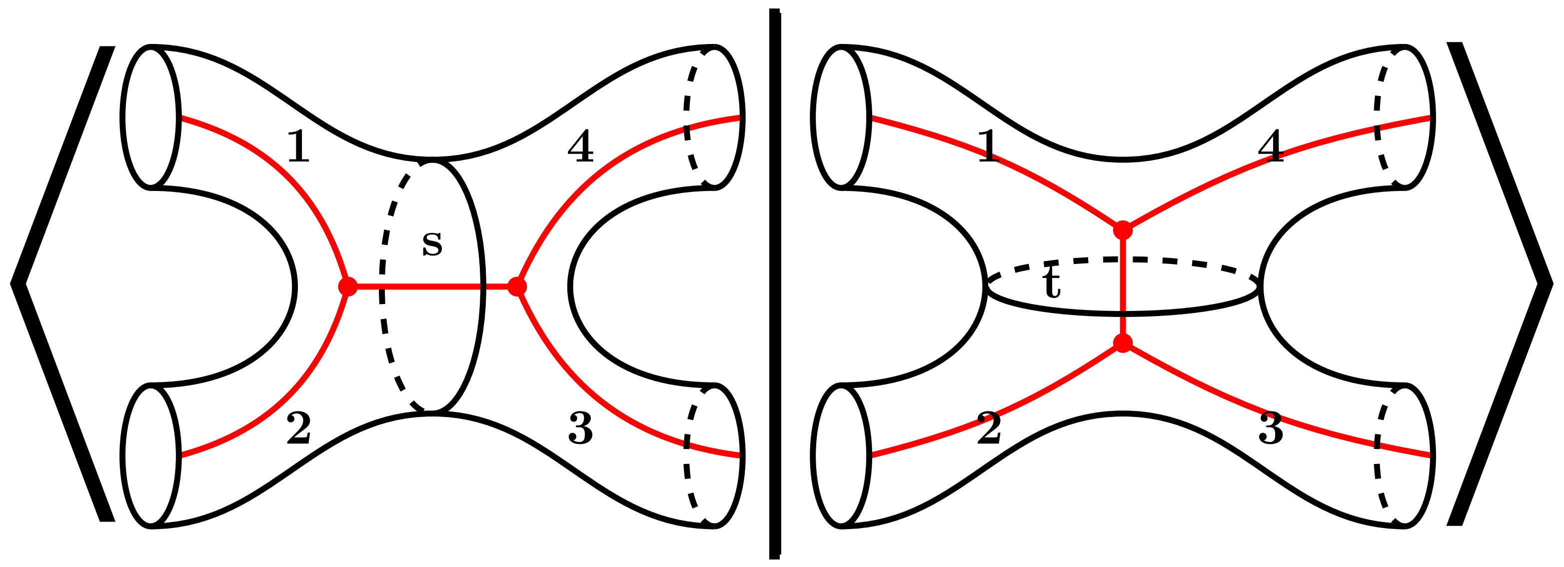


[Ponsot Teschner 1999]

- The crossing kernel $\mathbb{F}$ is remarkably known explicitly and satisfies the [Moore Seiberg 1989] consistency conditions [Teschner 2003]
- The inner product is preserved under crossing (can actually derive it this way)
  - Hilbert space closes under crossing and quantization is independent of basis
  - Enough data to define a consistent TQFT (à la modular tensor category)

For example

- The inner product of sphere four-point blocks in different channels computes the crossing kernel



$$= \frac{\mathbb{F}_{P_t P_s}\begin{bmatrix} P_1 P_2 \\ P_4 P_3 \end{bmatrix}}{\rho_0(P_s) C_0(P_1, P_2, P_s) C_0(P_3, P_4, P_s)}$$

$$= \frac{\begin{Bmatrix} P_1 P_2 P_s \\ P_3 P_4 P_t \end{Bmatrix}}{\sqrt{C_0(P_1, P_2, P_s) C_0(P_3, P_4, P_s) C_0(P_1, P_4, P_t) C_0(P_2, P_3, P_t)}}$$

"Virasoro 6j symbol"

Taking stock

$SL(2,\mathbb{R})^2$ Chern-Simons theory on initial value surface Σ

↓ seemingly arbitrary restriction on phase space

Teichmuller component $\mathscr{T}(\Sigma)$ of gauge theory phase space (hyperbolic metrics on Σ)

↓ quantization of Teichmuller space

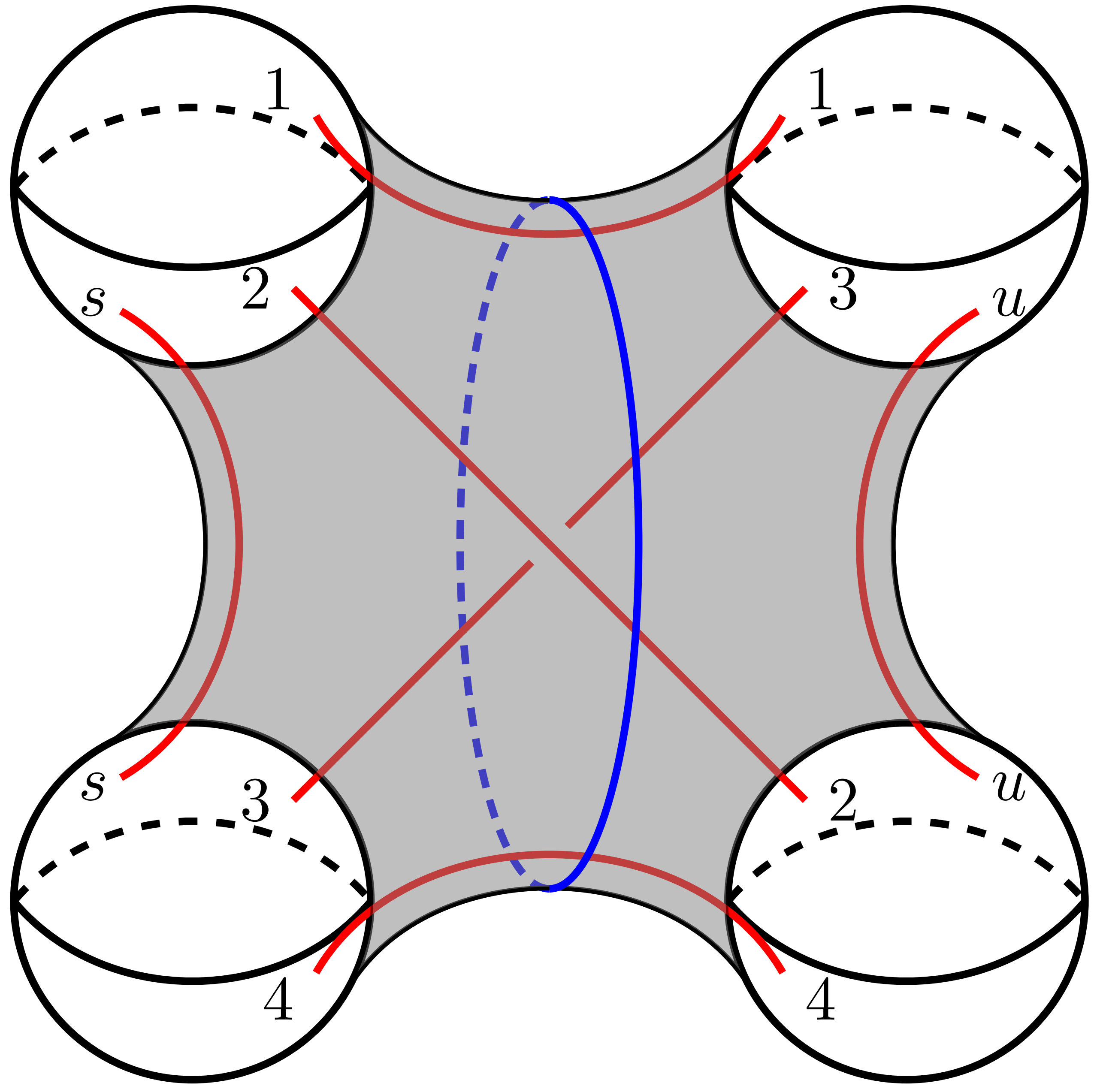
"Virasoro TQFT:" Hilbert space $\mathscr{H}(\Sigma)$ is the space of conformal blocks on Σ

Defines a consistent TQFT! (Despite some subtleties associated with infinite-dimensional Hilbert space)

Computing partition functions

- The role in life of a TQFT is to compute partition functions on three-manifolds
- If $\partial M = \sqcup_{i=1}^m \Sigma_i$, then $Z_{\text{Vir}}(M)$ should be thought of as a vector in the tensor product Hilbert space associated with the asymptotic boundaries

$$Z_{\text{Vir}}(M) \in \bigotimes_{i=1}^m \mathscr{H}(\Sigma_i)$$



- In practice, compute $Z_{\text{Vir}}(M)$ via **surgery techniques** (e.g. "Heegaard splitting"): $M \to M_1, M_2$

$$Z_{\text{Vir}}(M) = \langle Z_{\text{Vir}}(M_1) | Z_{\text{Vir}}(M_2) \rangle$$

- General prescription: surgery → insertion of complete set of states → inner product

Partition functions in 3d gravity

- The TQFT partition function is defined on a **fixed topology**.
  The gravity path integral involves a sum over topologies:

$$\sum_{\text{topologies M}} \int \frac{[dg]}{\text{Diff(M)}} e^{-S_{\text{grav}}[g]}$$

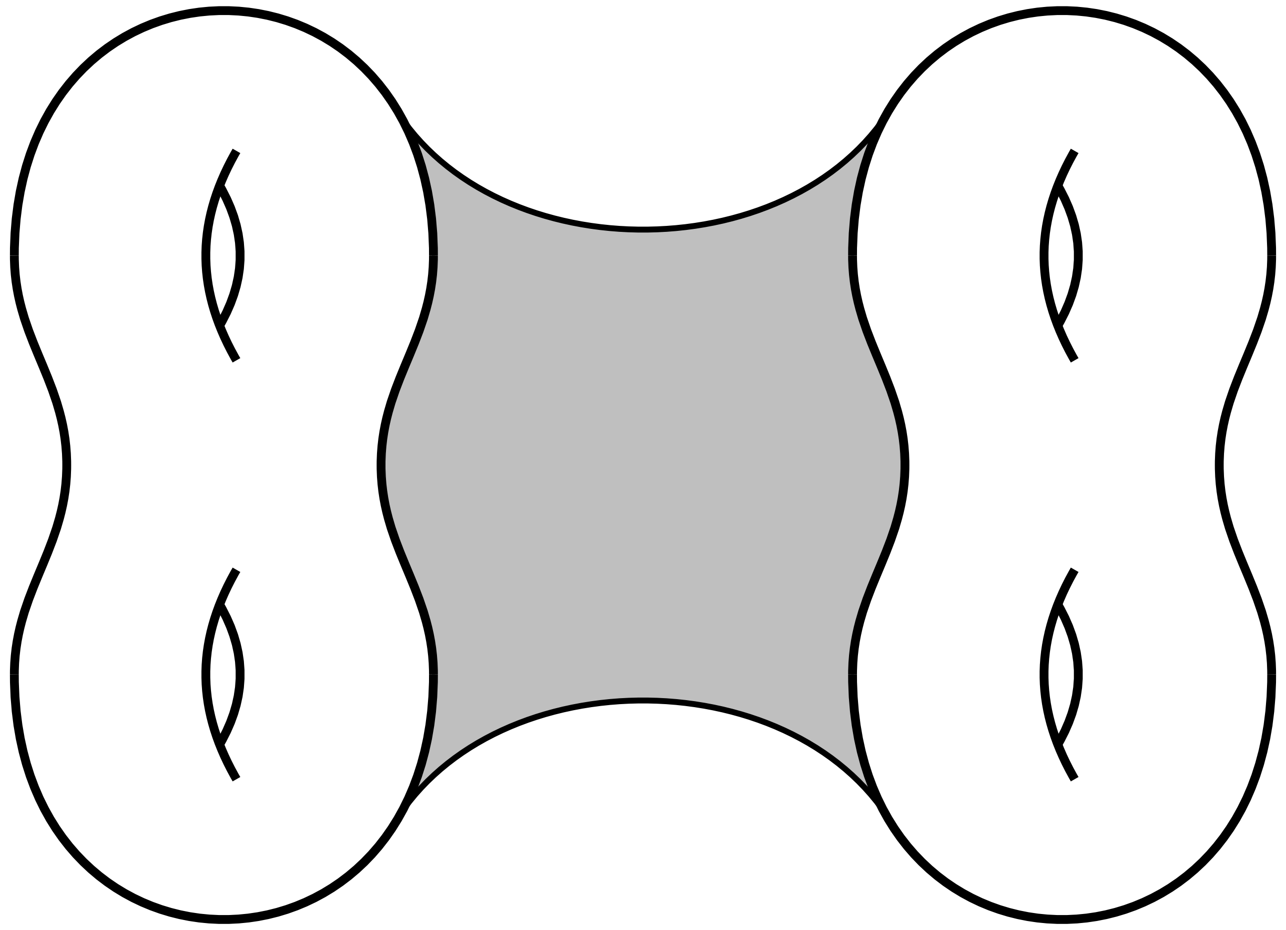
- **Mapping class group** $\text{Map}(M) = \text{Diff}(M)/\text{Diff}_0(M)$ captures the mismatch between the gravity and TQFT gauge groups

$$\Rightarrow \boxed{Z_{\text{grav}}(M) = \sum_{\gamma \in \text{Map}(\partial M)/\text{Map}(M)} |Z_{\text{Vir}}(\gamma \cdot M)|^2}$$

- Conjecture: $Z_{\text{Vir}}(M)$ finite for M a hyperbolic (a solution of Einstein equations)

Example: Euclidean wormhole

$M = \Sigma \times [0,1]$



Trivial state in TQFT: can glue it in without doing anything topologically

- Amounts to the **resolution of the identity** in the boundary Hilbert space $\mathscr{H}(\Sigma)$

$$Z_{\text{Vir}}(\Sigma \times [0,1]) = \int d\mathbf{P}\, \rho_\Sigma(\mathbf{P})\, |\mathscr{F}_\Sigma(\mathbf{P})\rangle \otimes |\mathscr{F}_\Sigma(\mathbf{P})\rangle$$

$$Z_{\text{Vir}}(\Sigma \times [0,1]; \mathbf{m}_1, \mathbf{m}_2) = Z_{\text{Liouville}}(\Sigma; \mathbf{m}_1, \mathbf{m}_2)$$

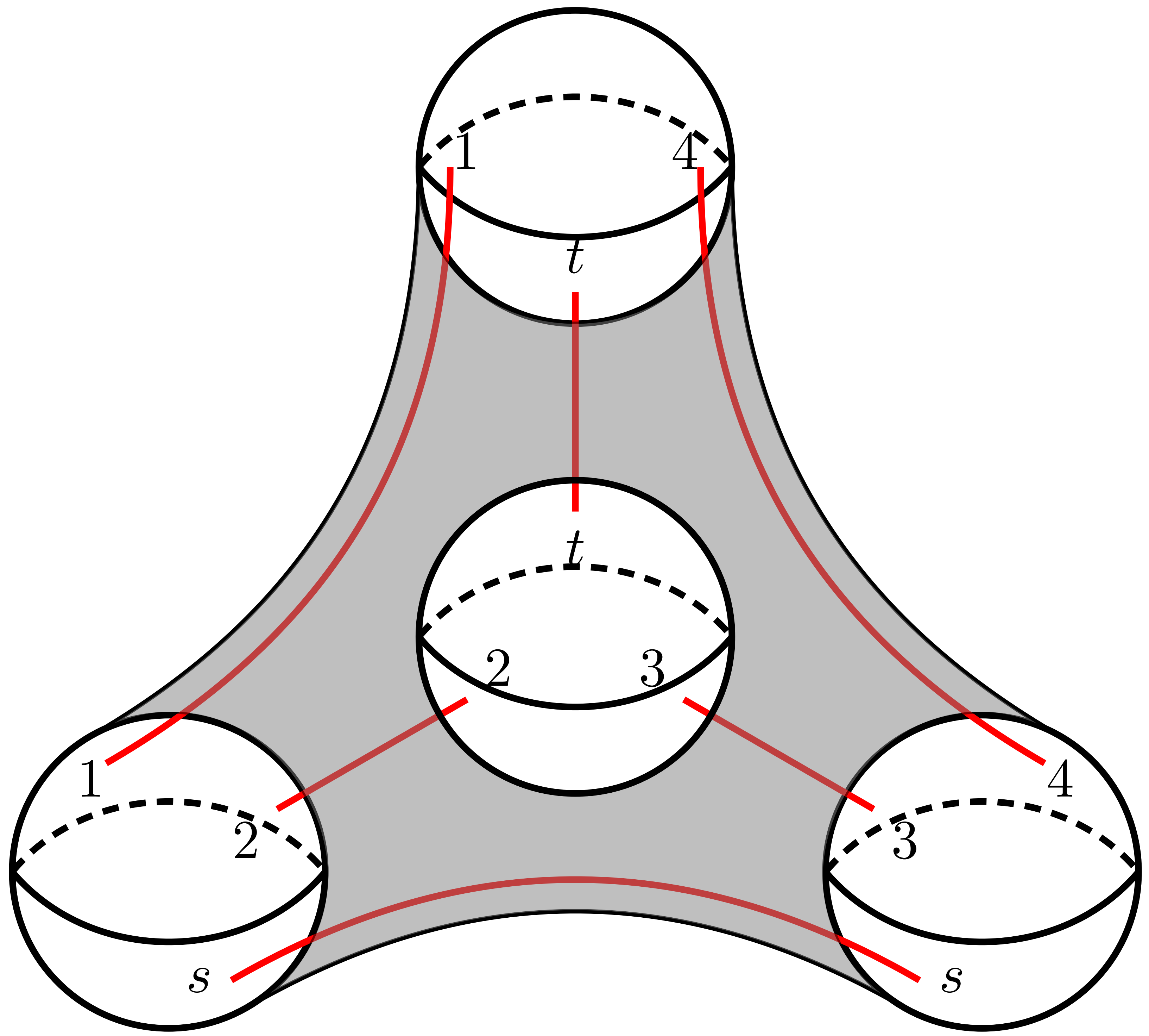
(Agrees with $\overline{Z_{\text{CFT}}(\Sigma; \mathbf{m}_1) Z_{\text{CFT}}(\Sigma; \mathbf{m}_2)}$ in the Gaussian ensemble)

- Full gravity partition function includes a sum over relative crossing transformations:

$$Z_{\text{grav}}(\Sigma \times [0,1]; \mathbf{m}_1, \mathbf{m}_2) = \sum_{\gamma \in \text{Map}(\Sigma)} \left| Z_{\text{Liouville}}(\Sigma; \mathbf{m}_1, \gamma \cdot \mathbf{m}_2) \right|^2$$

Example: four-boundary wormhole

- Leading non-Gaussianity in ensemble of CFT data controlled by four-boundary wormhole

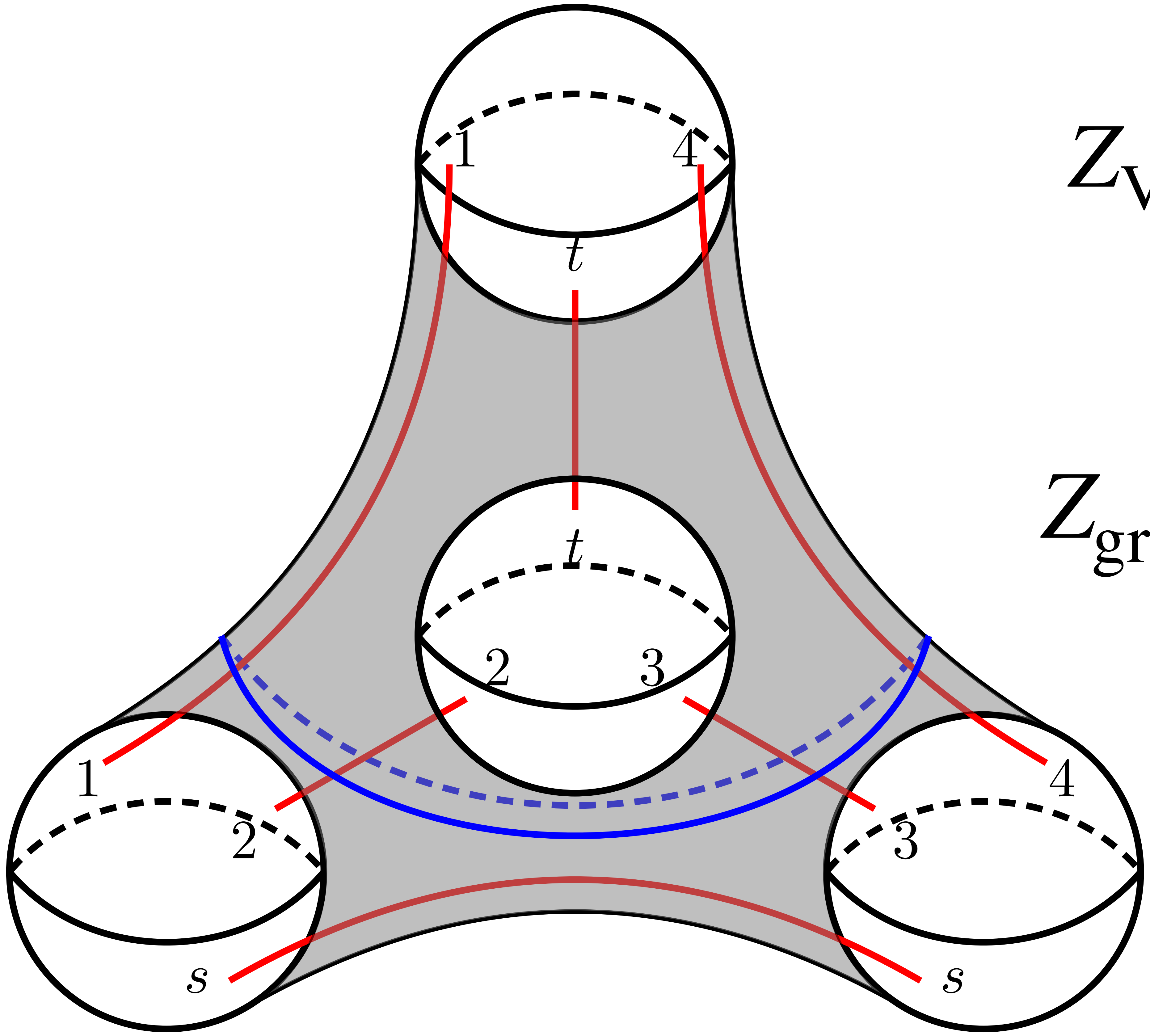


$$\longleftrightarrow \quad \overline{C_{12s}C_{34s}C_{14t}C_{23t}}$$

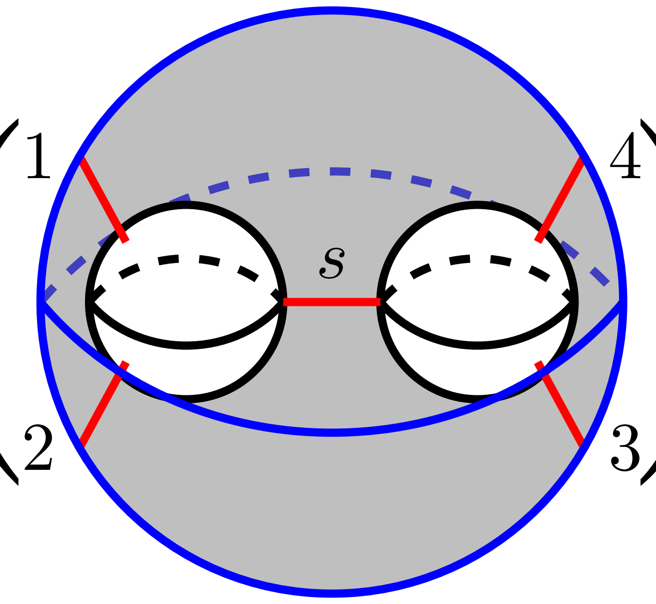
- Such non-Gaussianities are needed for consistency of the boundary ensemble description
- No known way to compute the gravitational action in the metric formalism

Example: four-boundary wormhole

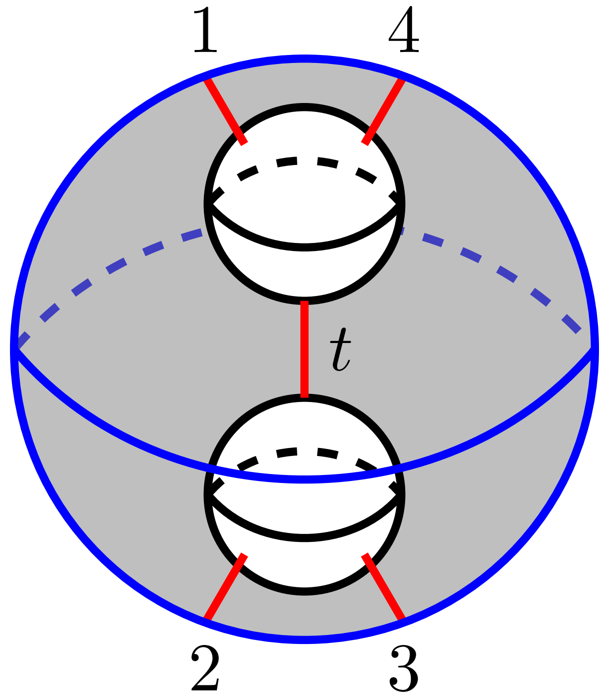
- Compute by splitting along a four-punctured sphere:



$$Z_{\mathrm{Vir}}(M) = \left\langle Z_{\mathrm{Vir}}\left(\right. \right.$$



$$\left. \right) \left| Z_{\mathrm{Vir}} \left(\right. \right.$$



$$\left. \left. \right) \right\rangle$$

$$Z_{\mathrm{grav}}(M) = \left| Z_{\mathrm{Vir}}(M) \right|^2 = \sqrt{\overline{C^2_{12s}}\,\overline{C^2_{34s}}\,\overline{C^2_{14t}}\,\overline{C^2_{23t}}}$$

$$\left| \begin{Bmatrix} P_1 P_2 P_s \\ P_1 P_4 P_t \end{Bmatrix} \right|^2$$

Virasoro 6j symbol

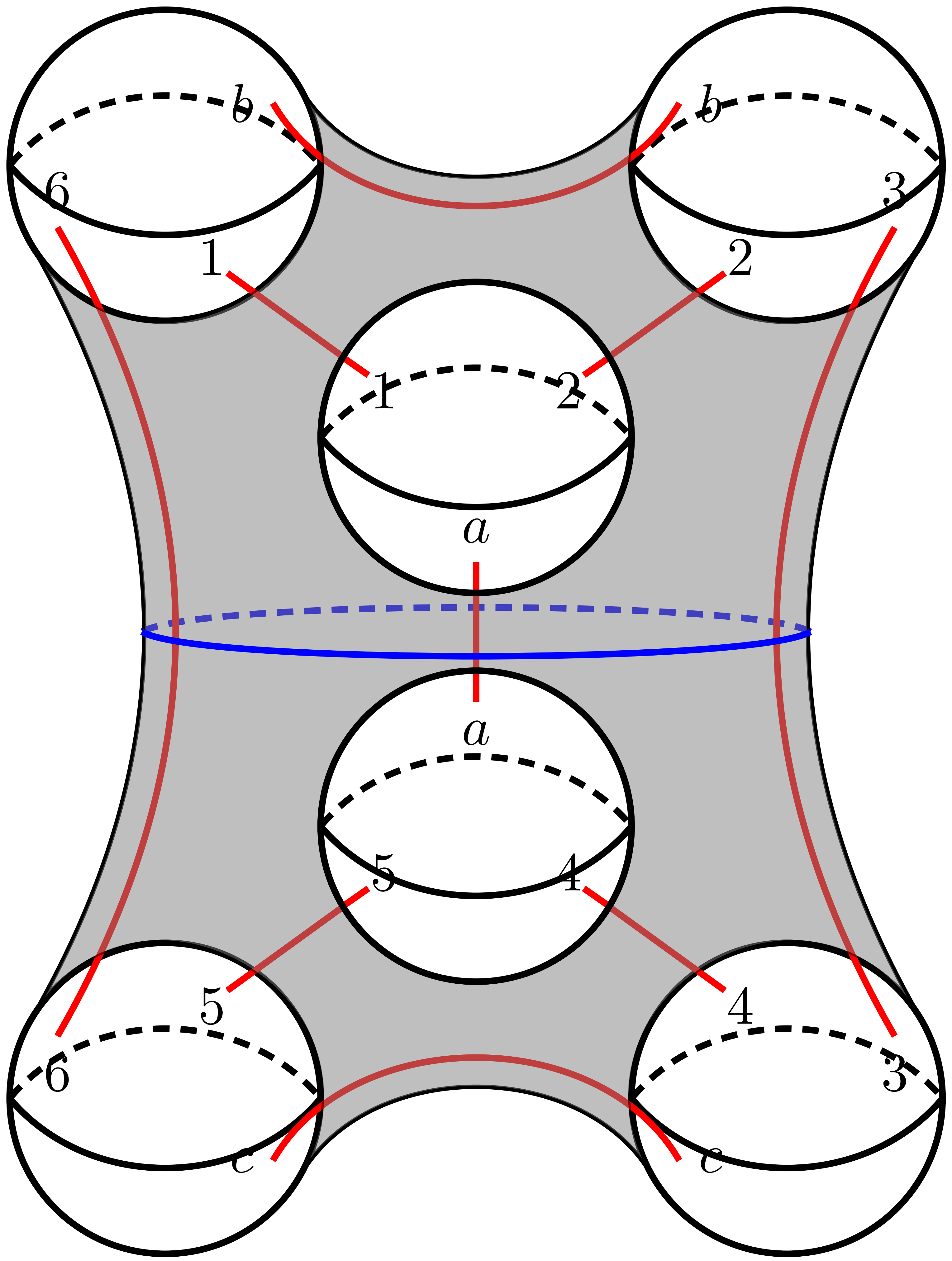
$c \gg 1$

$$\approx \exp\left(-\frac{c}{6\pi} \mathrm{vol(hyperbolic\ tetrahedron)} \right)$$

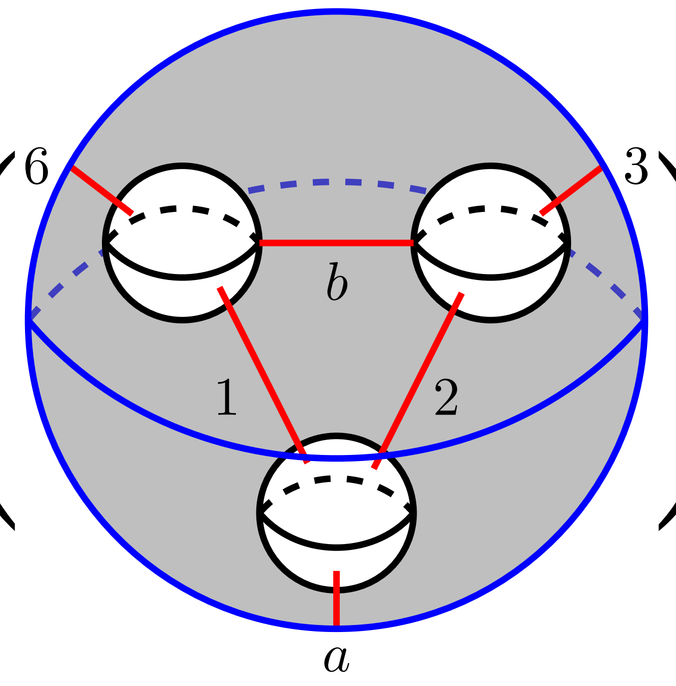
[Teschner Vartanov 2012]

Example: multi-boundary wormholes

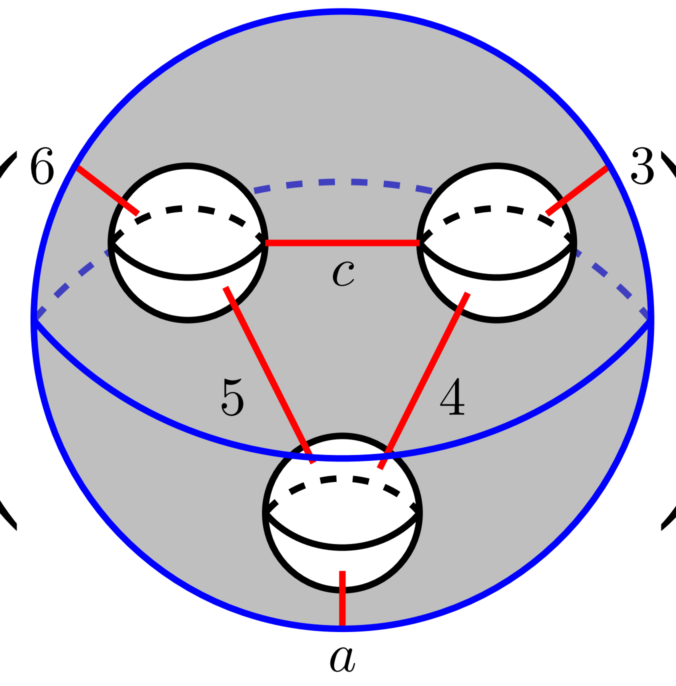
- Higher moments of CFT data: multi-boundary wormholes



$$Z_{\text{Vir}}(M) = \Big\langle Z_{\text{Vir}}\Big($$



$$\Big) \Big| Z_{\text{Vir}}\Big($$



$$\Big)\Big\rangle$$

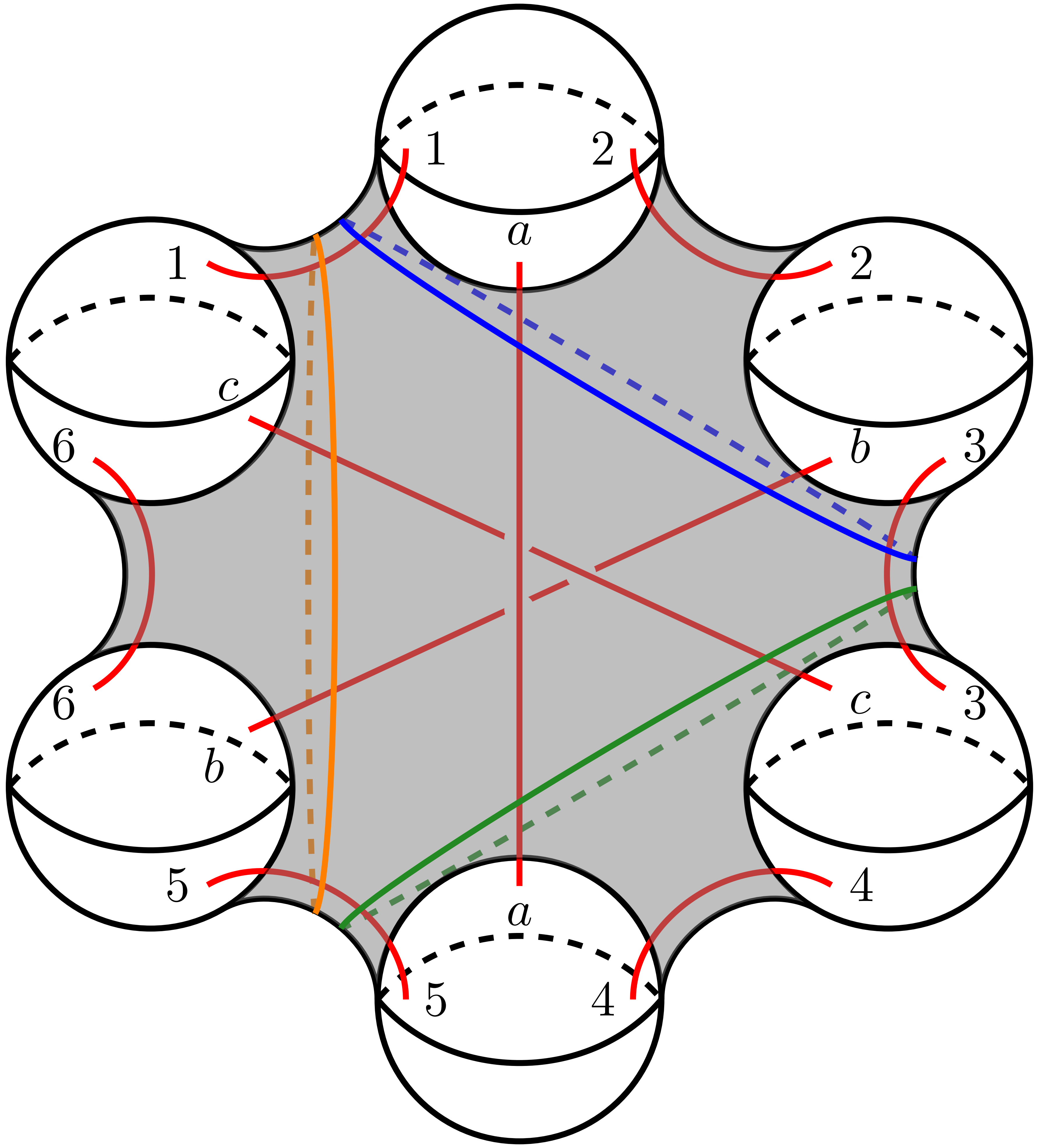
$$Z_{\text{grav}}(M) = \sqrt{\overline{C^2_{1a2}}\;\overline{C^2_{2b3}}\;\overline{C^2_{3c4}}\;\overline{C^2_{4a5}}\;\overline{C^2_{5c6}}\;\overline{C^2_{6b1}}}$$

$$\left| \begin{Bmatrix} P_1 P_2 P_a \\ P_3 P_6 P_b \end{Bmatrix} \begin{Bmatrix} P_3 P_4 P_c \\ P_5 P_6 P_a \end{Bmatrix} \right|^2$$

$$\longleftrightarrow \quad \overline{C_{1a2} C_{2b3} C_{3c4} C_{4a5} C_{5c6} C_{6b1}}$$

Example: multi-boundary wormholes

- Higher moments of CFT data: multi-boundary wormholes



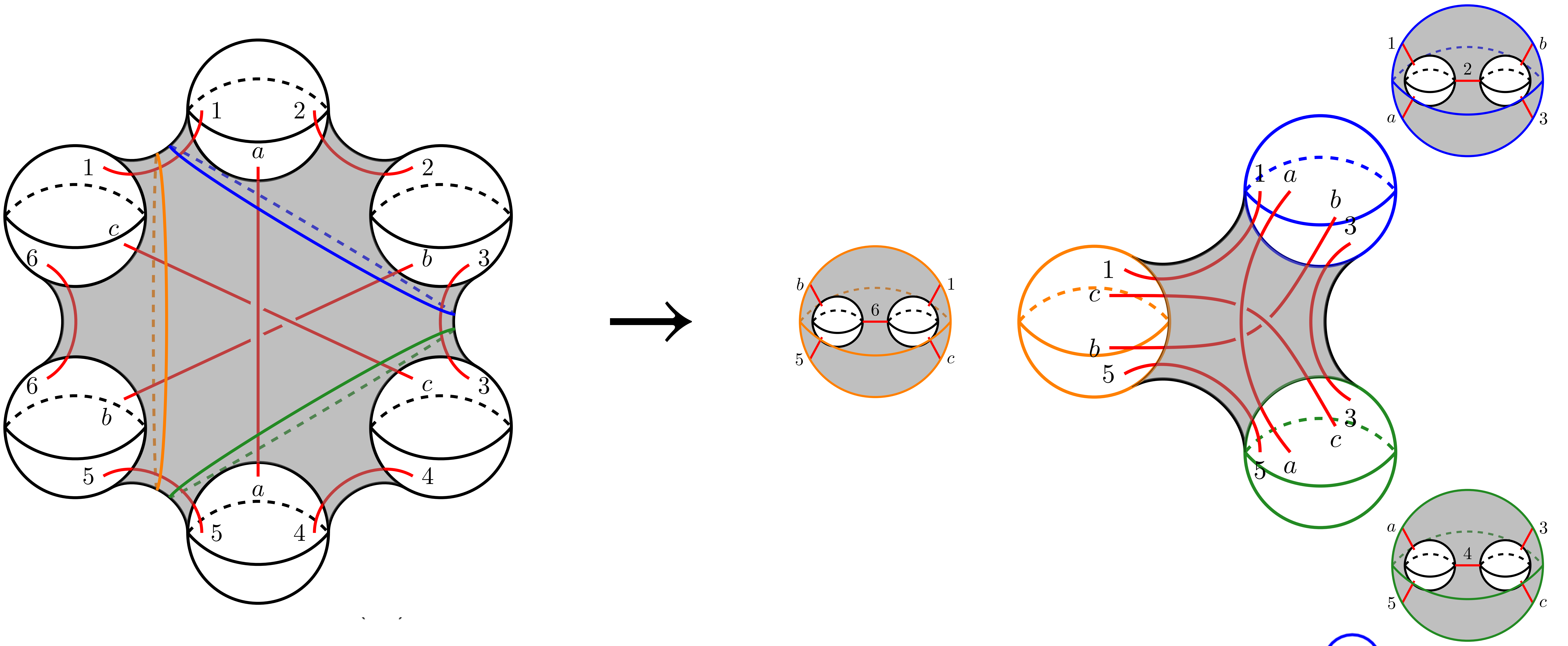
$$Z_{\text{grav}}(M) = (-1)^{\ell_1+\ell_3+\ell_5}\sqrt{\overline{C^2_{1a2}}\;\overline{C^2_{2b3}}\;\overline{C^2_{3c4}}\;\overline{C^2_{4a5}}\;\overline{C^2_{5b6}}\;\overline{C^2_{6c1}}}$$

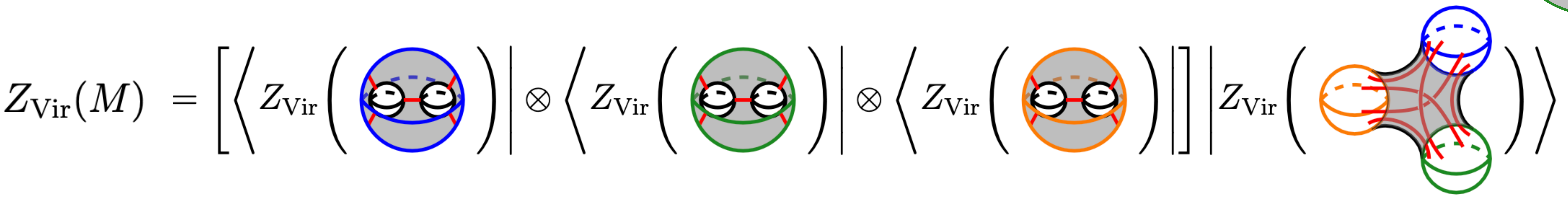
$$\left|\int dP_d\, e^{3\pi i P_d^2}\rho_0(P_d)\begin{Bmatrix} P_1 P_2 P_a \\ P_3 P_d P_b \end{Bmatrix}\begin{Bmatrix} P_3 P_4 P_c \\ P_5 P_d P_a \end{Bmatrix}\begin{Bmatrix} P_5 P_6 P_b \\ P_1 P_d P_c \end{Bmatrix}\right|^2$$

$$\longleftrightarrow \quad \overline{C_{1a2}C_{2b3}C_{3c4}C_{4a5}C_{5b6}C_{6c1}}$$

- Results suggest simple diagrammatic rules to compute statistics of CFT data:
  q-deformation of disk Feynman diagrams in JT gravity + matter [Jafferis Kolchmeyer Mukhametzhanov Sonner 2022]
- Simplicial 3d gravity??? 4-boundary wormhole as building block

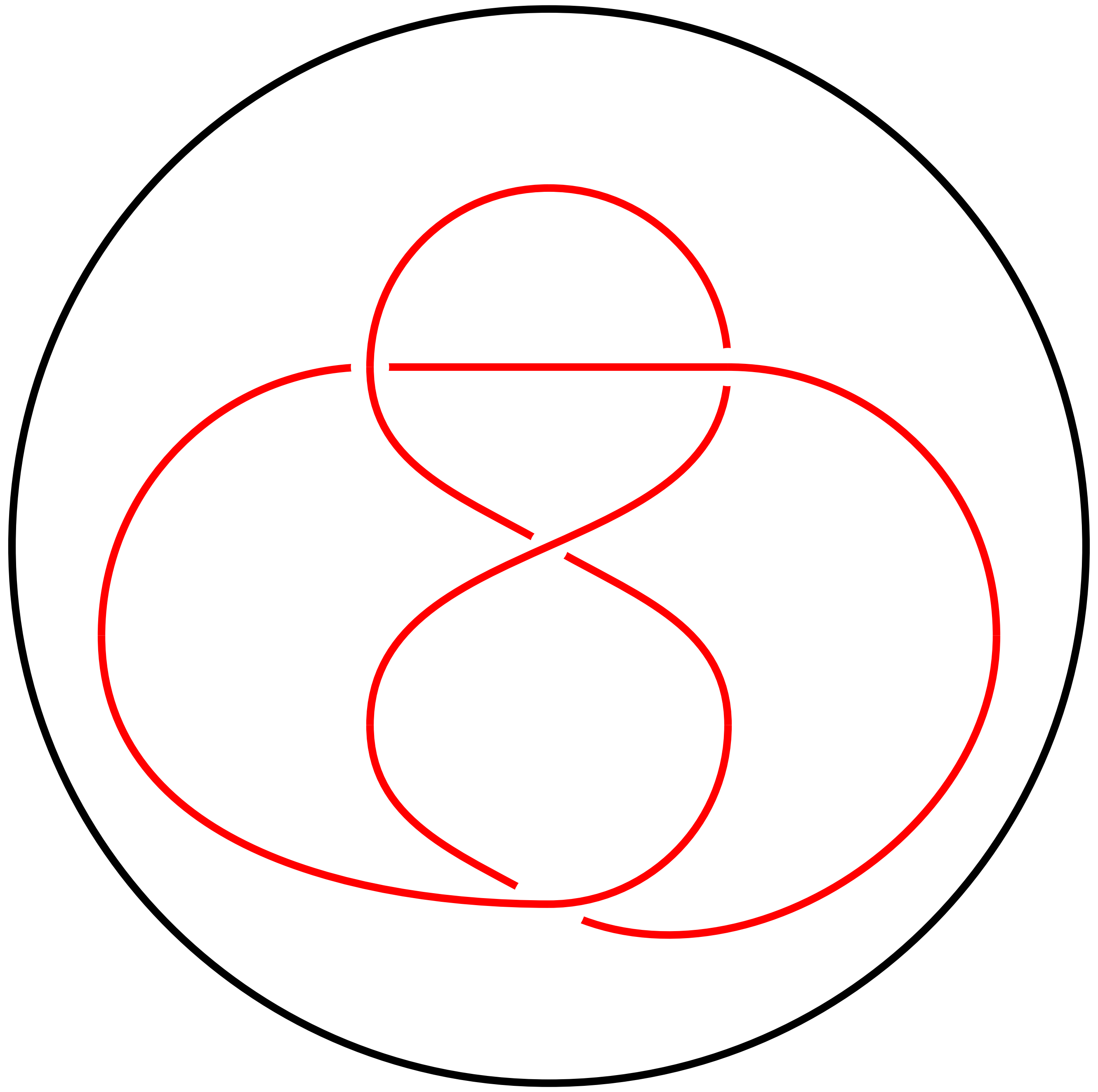
Heegaard splitting of the six-boundary wormhole





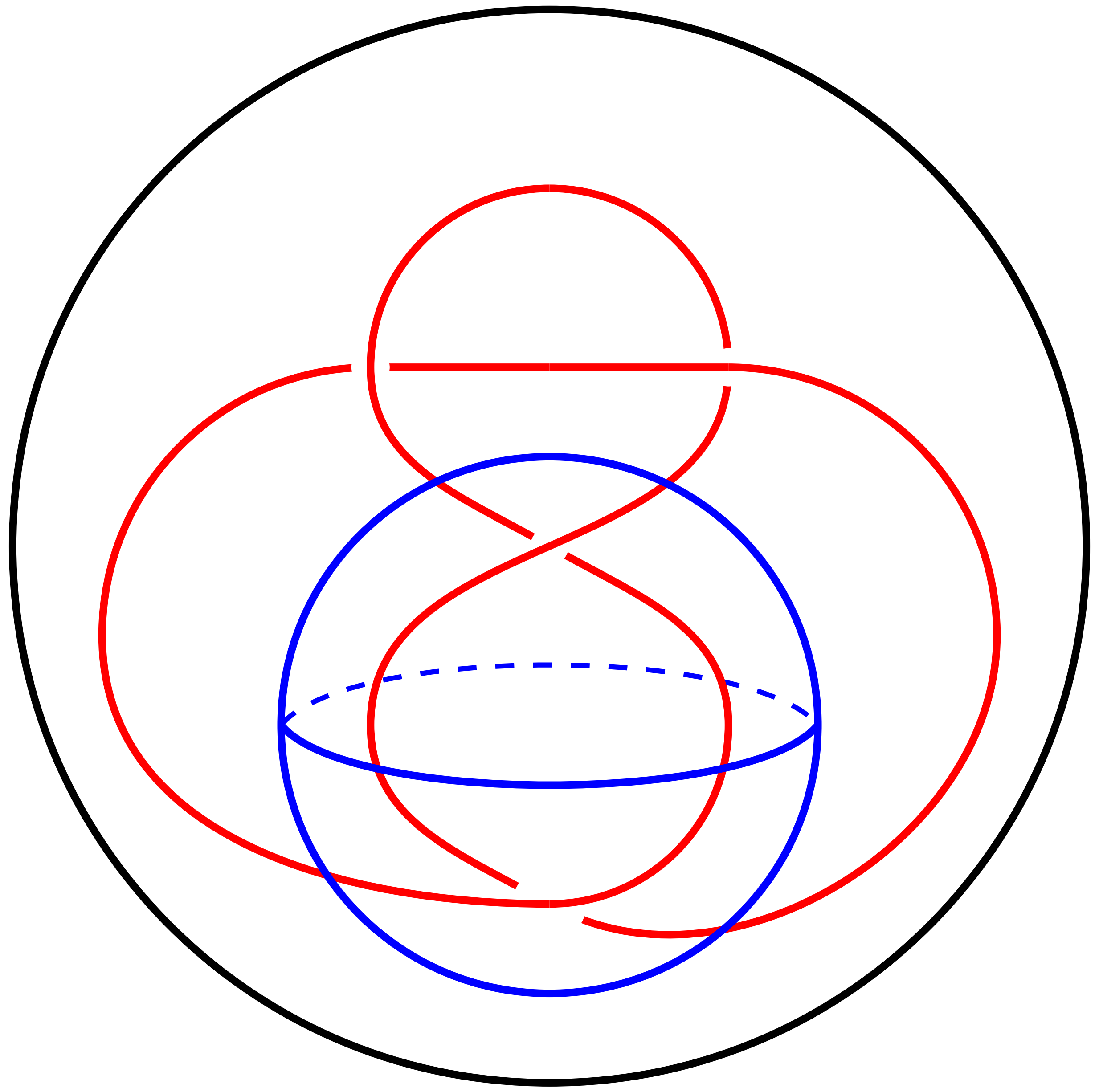
Example: hyperbolic links

- Virasoro TQFT partition functions on hyperbolic 3-manifolds define **topological invariants**. A zoo of possible examples to explore.
- For instance, hyperbolic "knot complements".
  Holographic interpretation unclear! (no asymptotic boundary)
- Classic example: **figure-eight knot complement**

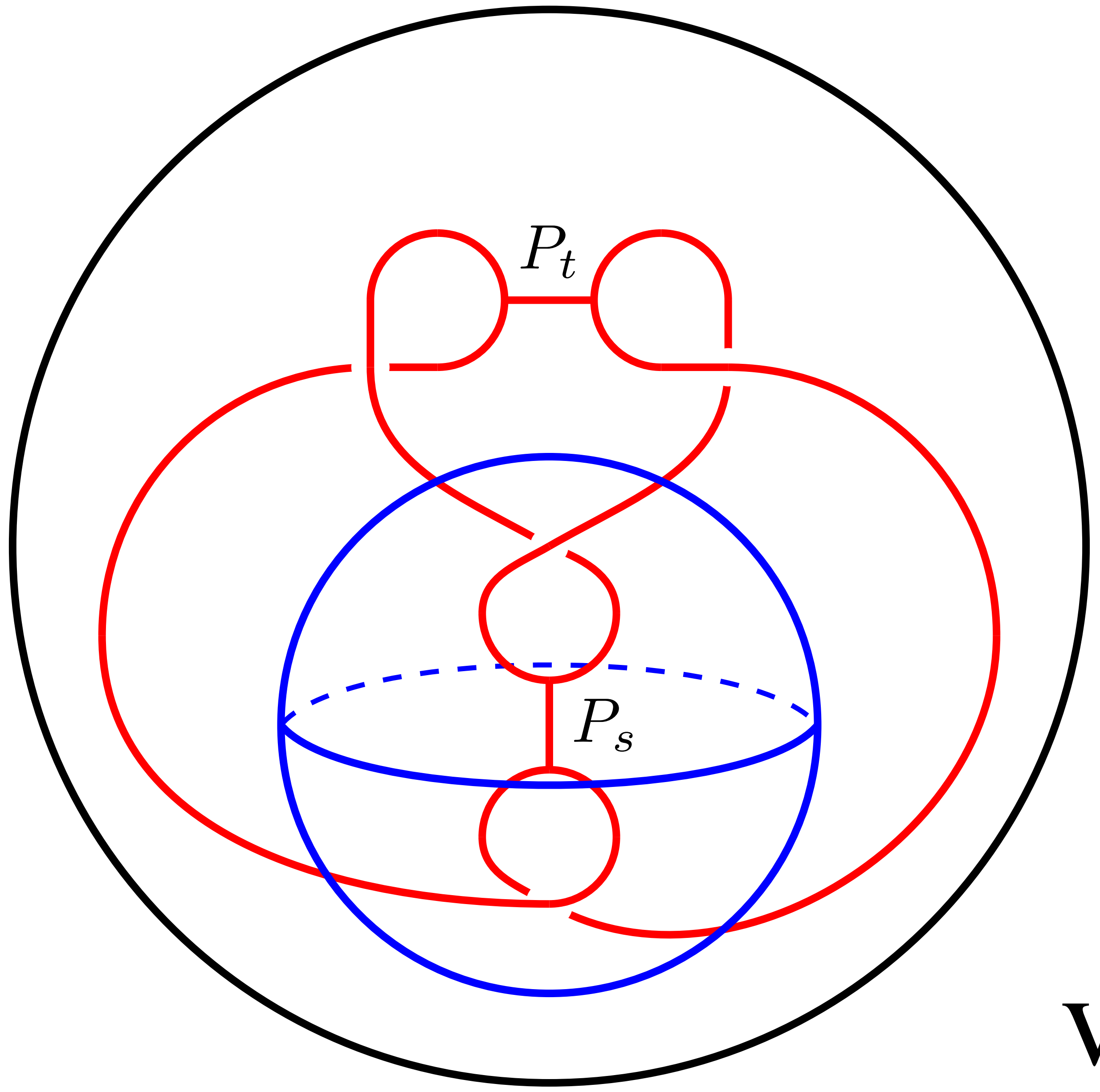


Example: hyperbolic links

- Compute by splitting along four-punctured sphere, doing some crossing transformations, then taking the inner product



$$\longrightarrow \int dP_s \, dP_t \, \mathbb{F}_{\mathbb{1}P_s} \mathbb{F}_{\mathbb{1}P_t}$$



vol(M)!

$$Z_{\text{Vir}}(M) = \int dP_s \, dP_t \, \rho_0(P_s)\rho_0(P_t) \, e^{2\pi i(P_t^2 - P_s^2)} \begin{Bmatrix} P & P & P_s \\ P & P & P_t \end{Bmatrix} \Bigg|_{P=0} \overset{c \gg 1}{\approx} \exp\left(-\frac{c}{12\pi} 2.02988\ldots\right)$$

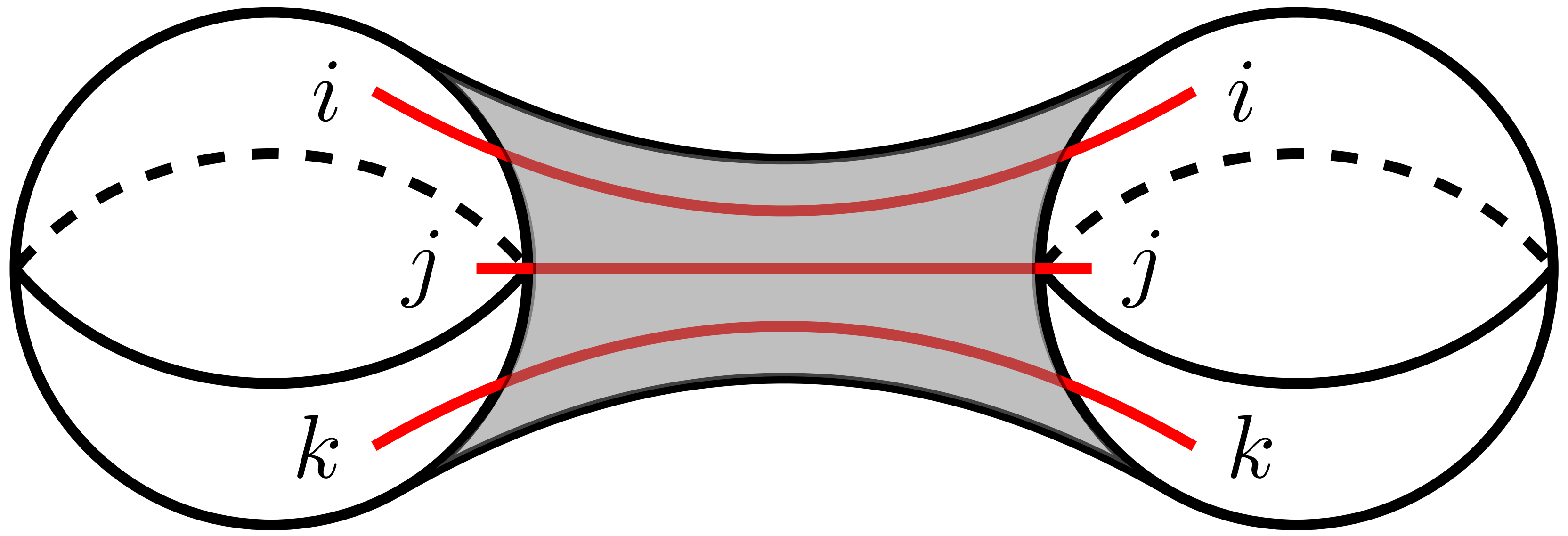
Discussion

- We have developed a proposal for AdS_3 quantum gravity on arbitrary hyperbolic 3-manifolds in terms of a novel but well-defined TQFT defined by the quantization of Teichmuller space
- Leads to a very **concrete and explicit prescription** to compute partition functions in 3d gravity using standard TQFT techniques
  - Trivializes or simplifies many computations that were burdensome or just not possible in the metric formulation of 3d gravity
  - Makes more manifest the relation with the CFT ensemble description of semiclassical 3d gravity, and provides an unambiguous setting for the gravitational determination of more nontrivial statistics of CFT data
- This technology should be instrumental in determining the extent to which any version of pure AdS_3 gravity makes sense as a quantum theory

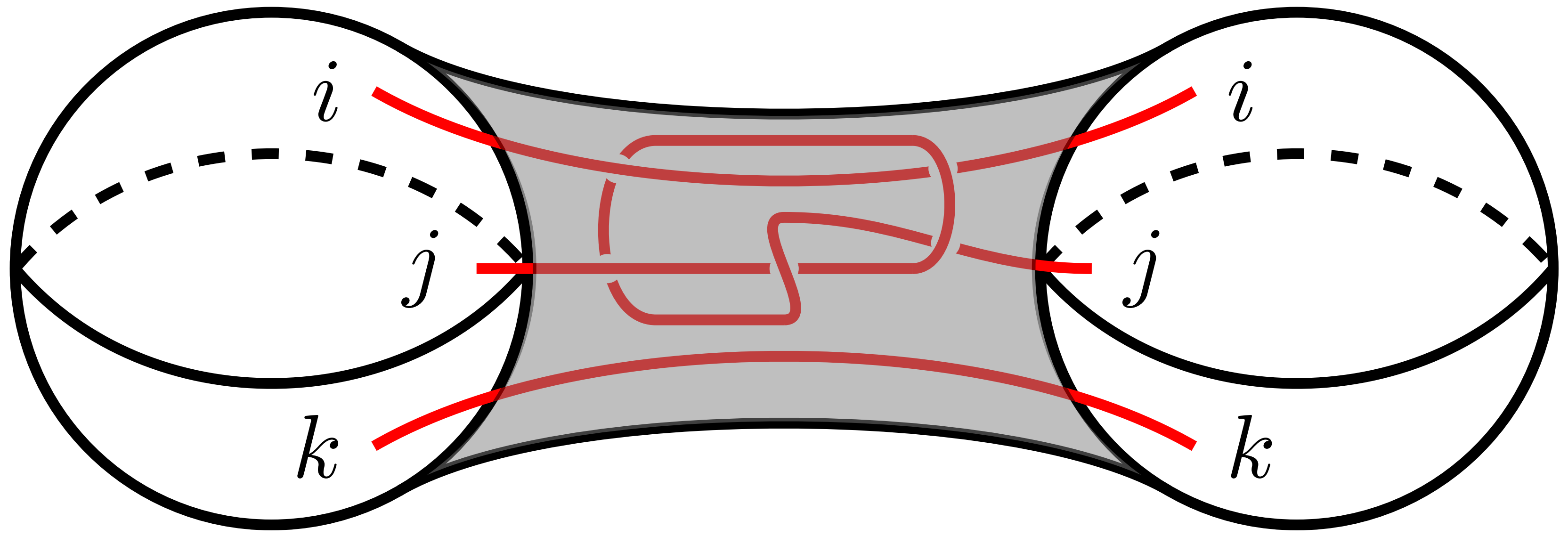
Future directions

- **Aspects of the boundary dual**
  - Relation between gravity partition functions and averaged CFT quantities vastly extended; begs for a **unifying description** that captures all higher moments
    - **Prove** the relationship between the gravity path integral and coarse-grained CFT quantities
    - Non-perturbative corrections to statistics of CFT data from **higher topologies** [WIP w/ Belin, Eberhardt, Liska, Post]
  - Sharpen **“random CFT”** [Cotler Jensen 2020] beyond Gaussian ensemble
    First steps taken in random tensor model of [Belin de Boer Jafferis Nayak Sonner 2023]
  - What ingredients need to be added to localize on to a specific exact solution of the crossing equations? (eg. [Benini Copetti di Pietro 2022])

$\overline{C_{ijk}^2} \supset$



$+$



$+ \dots$

Future directions

- **The sum over topologies**
  - Does the sum over hyperbolic topologies make sense?
  - Hyperbolic 3-manifolds (without asymptotic boundaries) are ordered by their volume.
    There are accumulation points in the spectrum of volumes (Dehn fillings of cusps), but they can arguably be dealt with (à la [Maloney Witten 2007])
  - The sum is presumably asymptotic. Signal of doubly non-perturbative effects in gravity?
- **Finiteness of the TQFT partition function and non-hyperbolic manifolds**
  - It would be nice to **prove** that $Z_{\text{Vir}}(M)$ is finite whenever M is hyperbolic
  - Similarly, what about **off-shell** contributions (non-hyperbolic three-manifolds) to the gravitational path integral?
    Do not obviously necessarily give ill-defined contributions to Z_{Vir}
  - A class of examples that would be nice to understand are provided by Seifert manifolds with torus boundary, cf. [Maxfield Turiaci 2020]. They admit a standard Heegaard splitting [Schultens 1995].
- **The semiclassical limit and a volume conjecture**
  - Can one prove the following refinement of the "volume conjecture" (cf. [Giombi Maloney Yin 2008])?

$$Z_{\text{Vir}}(M) = e^{-\frac{c}{6\pi}\text{vol}(M)}\Big[\prod_{\gamma\in\mathscr{P}}\prod_{m=2}^{\infty}\frac{1}{|1-e^{-m\ell(\gamma)}|^2}+O(c^{-1})\Big]$$

Future directions

- **Simplicial 3d gravity**
  - Some of our results hint towards a **simplicial formulation** of 3d gravity
  - This is actually an old idea; “state-sum” models [Ponzano Regge 1968; Boulatov 1992; …; Turaev Viro 1991; …; Andersen Kashaev 2011, 2013]
  - Similar comments have been made in the context of the tensor model of [Belin de Boer Jafferis Nayak Sonner 2023]
  - This is actually how we first tried to formulate the theory, but we failed for technical reasons related to the subtle aspects of VTQFT. Can a simplicial reformulation be made precise?

- **Generalizations and extensions**
  - Various amounts of supersymmetry, higher-spin symmetry, …
  - The construction described in this talk only works for $c = 3\ell/2G_N \geq 25$
  - Expect that $c \leq 1$ should be an equivalent/dual description (swap roles of timelike and spacelike Liouville)
  - If we complexify c away from these half-lines, the formulation of the TQFT is qualitatively different (presumably related to “de Sitter Liouville string” [WIP with Eberhardt Mühlmann Rodriguez])
  - e.g.: what is the Hilbert space for $c = 13 + i\mathbb{R}$?

Thank you!