Southeast University

October 31, 2025

Introduction to Q-cohomology and Fortuity

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Motivation

- The study of the supercharge Q-cohomology in supersymmetric field theories dates back to Witten's seminal paper: Constraints on supersymmetry breaking

- By Witten's argument, the Euler characteristic of the Q-cohomology, namely the Witten index, is independent of couplings; however, how the Q-cohomology itself depends on the couplings remains an open question.

Motivation

- It was conjectured that the spectrum of Q-cohomology classes in the $\mathcal{N} = 4$ super-Yang-Mills (SYM) is tree-level (classically) exact. [Kinney-Maldacena-Minwalla-Raju'05, Grant-Grassi-Kim-Minwalla'08, …]

- This non-renormalization conjecture opened a window for studying the microstates of black holes in the gravity dual of the $\mathcal{N} = 4$ SYM at strong 't Hooft coupling via constructing and manipulating the Q-cohomology classes at weak coupling, and motivated a series of recent works. [CC-Lin'22, Choi-Kim-Lee-Park'22, … many others]

Outline

- Introduction
- Q-cohomology, conjectures, and examples
- Q-cohomology in $\mathcal{N} = 4$ SYM, and the holographic dual
- S-duality test
- Conclusion

Q-cohomology, conjectures, and examples

BPS state/operators

- Consider the $\mathcal{N} = 2$ supersymmetry:

$$\{Q, Q^\dagger\} = H - E_{\text{BPS}} \equiv \Delta\,, \quad Q^2 = 0 = Q^{\dagger 2}$$

- The BPS states $|\Psi\rangle$: $\quad H|\Psi\rangle = E_{\text{BPS}}|\Psi\rangle \quad \Leftrightarrow \quad Q|\Psi\rangle = 0 = Q^\dagger|\Psi\rangle$

  (BPS bound: $E \geq E_{\text{BPS}}$)

- Standard Hodge theory argument:

$$\text{BPS states} \longleftrightarrow Q\text{-cohomology} \qquad \frac{\left\{|\Psi\rangle \,\middle|\, Q|\Psi\rangle = 0\right\}}{\left\{|\Psi\rangle \,\middle|\, |\Psi\rangle = Q|\Psi'\rangle\right\}}$$

Witten index

- The Witten index is the Euler characteristic of the Q-cohomology

$$I = \mathrm{Tr}\,(-1)^F e^{-\beta\Delta} = \mathrm{Tr}_{\mathrm{BPS}}\,(-1)^F = \mathrm{Tr}_{\text{Q-coho}}\,(-1)^F$$

- Witten argued that:

  1. I is independent of β

  2. I is independent of coupling constants in H, (as long as the Hilbert space is unchanged).

- How does the Q-cohomology (BPS spectrum) depend on the couplings?

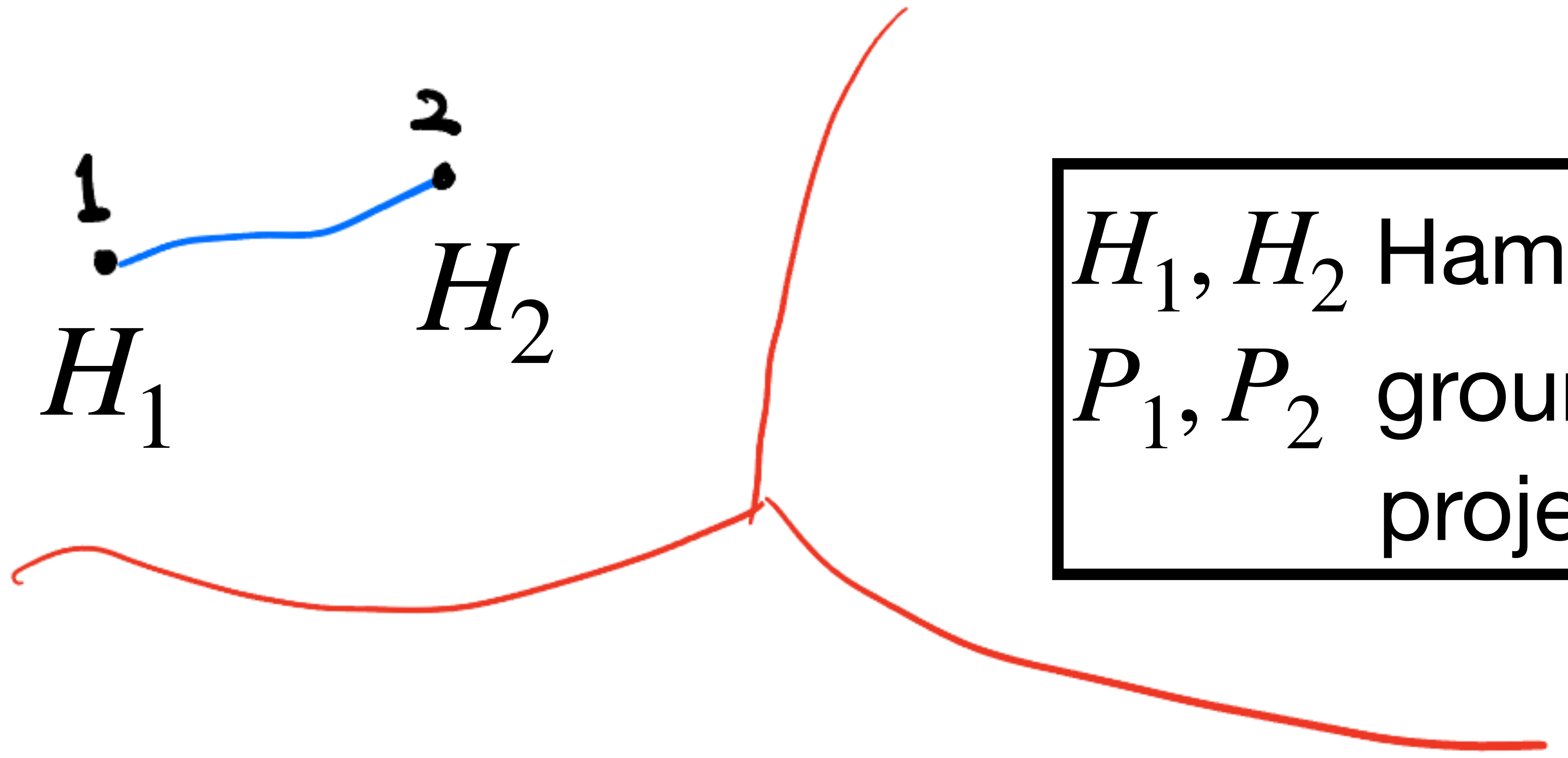
Quantum phase transition

- In CMT, one studies the space of Hamiltonians.
- Two points, 1 and 2, in this space belong to the same phase if there exists a path from 1 to 2 such that the gap above the ground states does not close, i.e., the Hilbert space of the ground states is preserved.

$$P_2 = UP_1U^\dagger$$

$U = Pe^{i\int F(s)ds}$ a local unitary

(In general, $H_2 \neq UH_1U^\dagger$)



H_1, H_2 Hamiltonians
P_1, P_2 ground space projectors

“Phases” of SUSY theories

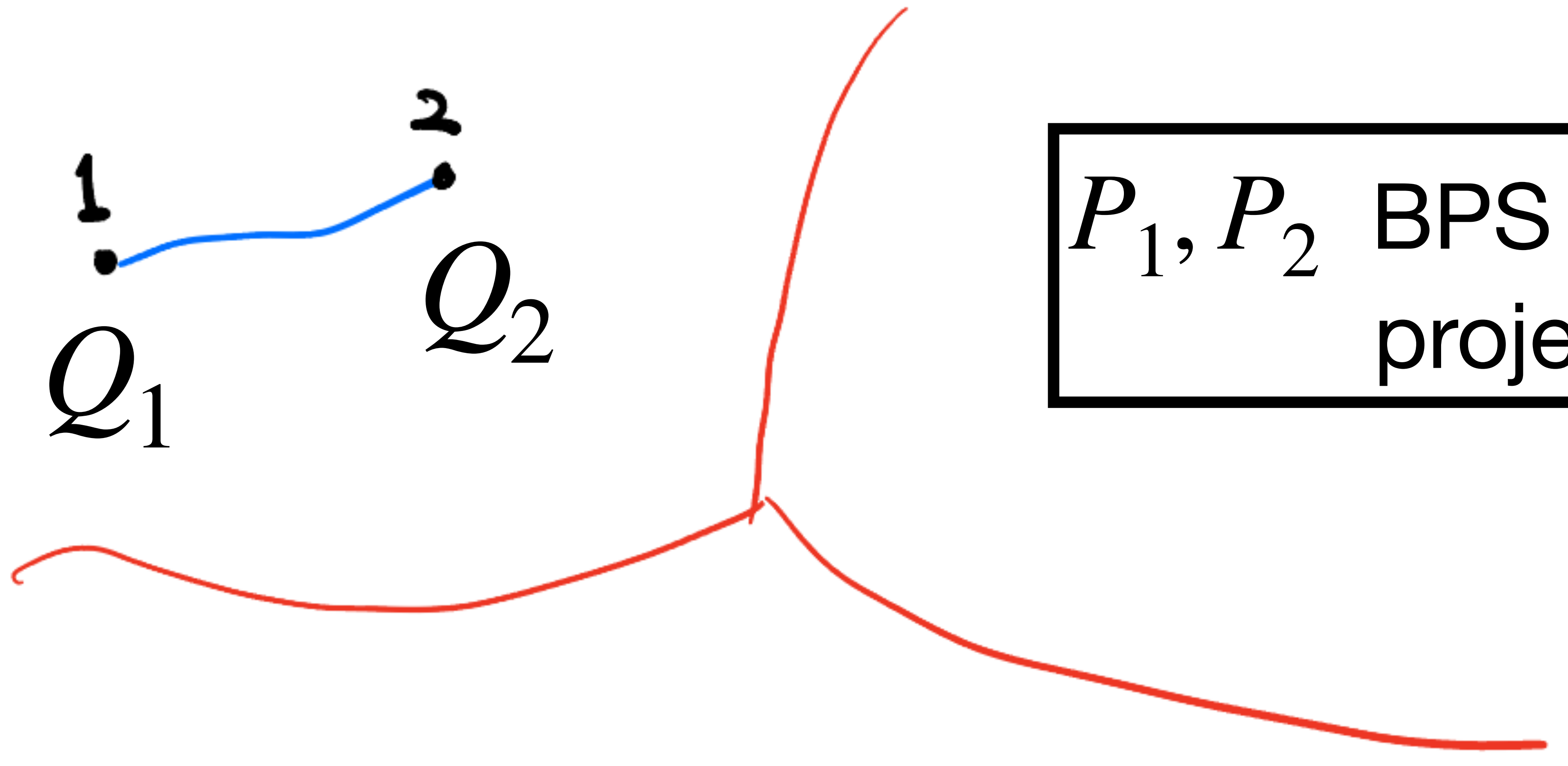
- Let us consider the space of supercharges.

- **Definition:** Q_1 and Q_2 are in the same phase if ∃ a path from 1 to 2, s.t. the Q-cohomology is preserved, i.e., the BPS subspace is preserved.

$$Q_2 = MQ_1M^{-1}$$

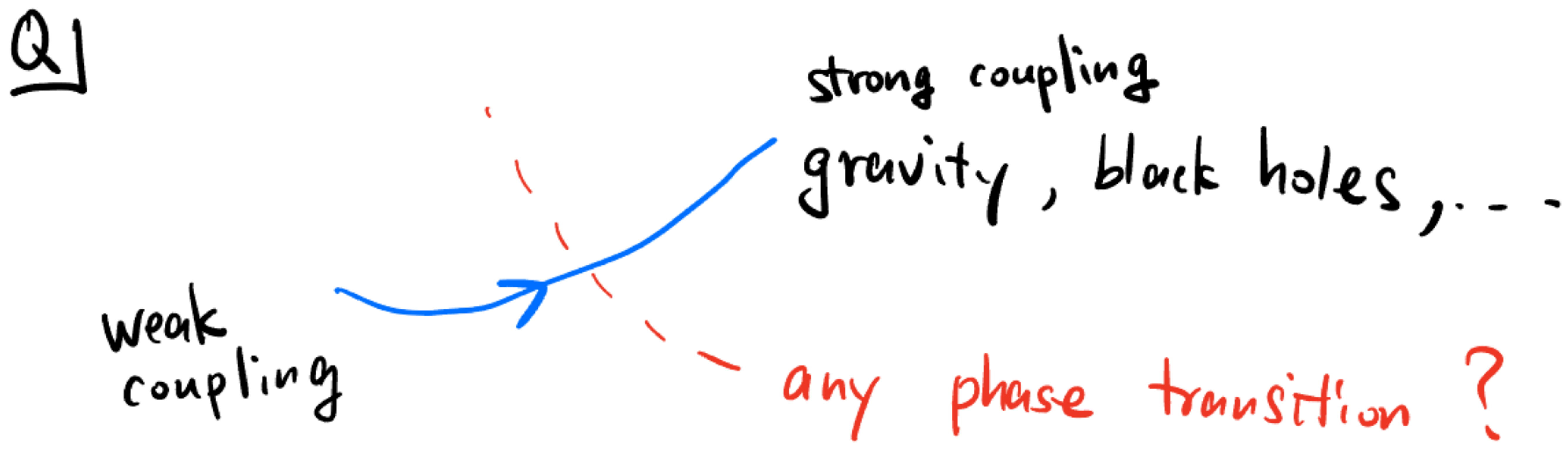
$M = Pe^{\int_1^2 F(s)ds}$ a local invertible

M might not preserve BPS subspace, $P_2 \neq MP_1M^{-1}$.



P_1, P_2 BPS subspace projectors

Non-renormalization conjectures



- Weak conjecture: There is no (codimension one) phase boundary. Phase transition can only occur at discrete points, usually free points.

- Strong conjecture: The Q-cohomology is tree-level (classically) exact.

$$\{Q_{\text{tree}}, Q^\dagger_{\text{tree}}\} = \Delta_{1-\text{loop}},$$ so the BPS spectrum is 1-loop exact.

Example 1: $\mathcal{N} = 2$ SYK model

- $\mathcal{N} = 2$ SYK model: complex fermions ψ_i for $i = 1, \cdots, N$ [Fu-Gaiotto-Maldacena-Sachdev'16]

$$Q = \sum_{i, \cdots, i_q = 1}^{N} C_{i_1 \cdots i_q} \psi^{i_1} \cdots \psi^{i_q}$$

- The states are **constant** differential forms:

$$|\alpha\rangle = \frac{1}{p!} \sum_{i_1, \cdots, i_p = 1}^{N} \alpha_{i_1 \cdots i_p} \psi^{i_1} \cdots \psi^{i_p} |\Omega\rangle \ , \quad \psi^i \leftrightarrow dx^i \ , \quad (\alpha \text{ is a } p\text{-form})$$

- Q acts as a wedge product: $\quad \alpha \mapsto C \wedge \alpha \quad$ (C is a q-form)

Example 1: $\mathcal{N} = 2$ SYK model

- The BPS spectrum (Q-cohomology) is invariant under generic deformation of coupling $C_{i_1 \cdots i_q}$. Both the weak and strong conjectures are true in this example.

- A side comment: The BPS spectrum exhibits a very interesting **R-charge concentration** property — the BPS states in a cochain complex all have the same R-charge.

$$\cdots \xrightarrow{Q} H^{n_c - 1} \xrightarrow{Q} \color{red}{H^{n_c}} \xrightarrow{Q} H^{n_c + 1} \xrightarrow{Q} \cdots$$

- This property is closely related to the low-energy supersymmetric JT. [Chang-Chen-Sia-Yang'24]

Example 2: Sigma model

- Consider a supersymmetric particle on a manifold $\mathscr{M}$ with coordinates x^i and superpartners ψ^i. The states are differential forms:

$$|\alpha\rangle = \frac{1}{p!} \sum_{i_1,\cdots,i_p=1}^{N} \alpha_{i_1\cdots i_p}(x)\,\psi^{i_1}\cdots\psi^{i_p}\,|\Omega\rangle$$

- The supercharge $Q = p_i\psi^i = dx^i \frac{\partial}{\partial x^i} = d$ is the de Rham differential.

- Q-cohomology = de Rham cohomology

Adding superpotential

- Adding a superpotential $h(x)$ to the system, the supercharge becomes

$$Q = \left(g_{ij}\dot{x}^j + i\frac{\partial h}{\partial x^i} \right) \psi^i = e^{-h} Q_{h=0} e^h$$

- The Q-cohomology is independent of deformations of h and the metric g_{ij} of the manifold, as long as the manifold is compact and smooth, and the superpotential is finite.

- The weak conjecture is true.

Strong conjecture?

- Consider perturbation theory around the free points (large mass limit), i.e., around each critical point $\partial h / \partial x^i = 0$

$$h = m_{ij} x^i x^j + O(x^3) \ , \quad \text{E.V.}(m_{ij}) \gg 1.$$

- For each critical point, there is one BPS state whose form-degree (fermion number) equals the number of negative eigenvalues of m_{ij} (the Morse index).

- These BPS states may receive instanton corrections and get lifted.

- The strong conjecture is not true.

Expample 3: D1-D5 CFTs

- For CFTs, there are usually two choices of quantization.
  1. On $\mathbb{R}^{1,d-1} \longrightarrow$ a continuous spectrum. Usually, the ground is inside the continuum. If the ground state is separated by a gap, then the theory is topological.
  2. On $S^{d-1} \times \mathbb{R} \longrightarrow$ a discrete spectrum (for compact CFT). The states correspond one-to-one to local operators on $\mathbb{R}^d$.

- We would consider the $S^{d-1} \times \mathbb{R}$ case in which $Q^\dagger = S$ (a conformal supercharge).

- The space of Q is the superconformal manifold.

Expample 3: D1-D5 CFTs

- In the superconformal manifold of the D1-D5 CFTs, there is a special point that the theory is described by a symmetric orbifold

$$\mathrm{Sym}^N(M_4) \qquad \text{for } M_4 = T^4 \text{ or } K3$$

- We study the conformal perturbation theory around this orbifold point.

- For $N = 2$, up to the order we computed, the Q-cohomology under the first-order deformation exactly matches the known exact BPS partition function.

- This provides evidence for the strong conjecture in this case.

$\mathcal{N} = 4$ SYM

BPS operators in $\mathcal{N} = 4$ SYM

- State/operator correspondence: $O \leftrightarrow |O\rangle$
- $\mathcal{N} = 4$ SYM has 16 supercharges and 16 conformal supercharges
- Pick one supercharge $Q \equiv Q_-^4$ and one conformal supercharge $S = Q^\dagger$
- Supersymmetry algebra:
  $\Delta \equiv 2\{Q, Q^\dagger\} = D - J_1 - J_2 - q_1 - q_2 - q_3 \geq 0,$ D : dilatation
- $1/16$-BPS operators: O with $\Delta = 0 \Leftrightarrow QO = 0 = Q^\dagger O$

BPS operators in $\mathcal{N} = 4$ SYM

- Consider $\mathcal{N} = 4$ SYM with $\mathrm{U}(N)$ gauge group.

- All operators are $\mathrm{U}(N)$ invariant composites of fundamental fields with covariant derivatives.

- Fundamental fields and derivatives (letters):

$$N \times N \text{ matrix:} \quad \Phi^{[IJ]}\,, \quad \Psi_{I\alpha}\,, \quad \overline{\Psi}^{I}_{\dot{\alpha}}\,, \quad A_\mu\,, \quad D_\mu = \partial_\mu - iA_\mu$$

$$\mathrm{SU}(4)_R : \; I = 1,\cdots,4\,, \quad \mathrm{SO}(1,3) : \; \mu = 0,\cdots,3, \quad \mathrm{SU}(2)\times\mathrm{SU}(2) : \; \alpha, \dot{\alpha} = \pm$$

BPS Letters

- **BPS letters** ($\Delta = 0$):
  - Fields: $\phi^i \equiv \Phi^{4i}\,, \quad \psi_i \equiv -i\Psi_{i+}\,, \quad \lambda_{\dot{\alpha}} \equiv \overline{\Psi}^4_{\dot{\alpha}}\,, \quad f \equiv F_{\mu\nu}(\sigma^{\mu\nu})_{++}$
  - Derivatives: $D_{\dot{\alpha}} \equiv (\sigma^\mu)_{+\dot{\alpha}} D_\mu \qquad (i = 1,2,3)$
- **BPS superfield** (a generating function) with auxiliary variables $(z^+, z^-, \theta_1, \theta_2, \theta_3)$: $z^\pm$ commuting, θ_i anti-commuting variables [Grant-Grassi-Kim-Minwalla'08, CC-Yin'13]

$$\Psi(z^+, z^-, \theta_1, \theta_2, \theta_3) = -i \sum_{n=0}^{\infty} \frac{(z^{\dot{\alpha}} D_{\dot{\alpha}})^n}{n!} \left[\frac{z^{\dot{\beta}} \lambda_{\dot{\beta}}}{n+1} + 2\theta_i \phi^i + \epsilon^{ijk} \theta_i \theta_j \psi_k + 4\theta_1\theta_2\theta_3 f \right]$$

- It satisfies $\Psi(z^\alpha, \theta_i)\,|_{z^\alpha=0, \theta_i=0} = 0.$

Superconformal manifold

- The superconformal manifold of the $\mathcal{N}=4$ SYM is parametrized by the complexified coupling

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{\mathrm{YM}}^2}$$

- The free point is at $\tau = i\infty$, where any gauge invariant made out of Ψ is a BPS operator, because $Q\Psi = 0$.

- For example:

$$\mathrm{Tr}[\psi_1\phi_1]\,\mathrm{Tr}[\lambda_+ D_1 D_2 f] = \mathrm{Tr}[\partial_{\theta_2}\partial_{\theta_3}\Psi\partial_{\theta_1}\Psi]\,\mathrm{Tr}[\partial_{z^+}\Psi\partial_{z^+}\partial_{z^-}\partial_{\theta_1}\partial_{\theta_2}\partial_{\theta_3}\Psi]\Big|_{z^\alpha=\theta_i=0}$$

Tree-level cohomology

- At the tree-level (classical), the Q-action becomes

$$Q(\Psi) = \Psi^2$$

and satisfies the Leibniz rule: $Q(AB) = Q(A)B + (-1)^{|A|}AQ(B)$.

- The tree-level cohomology differs from the free cohomology.

- For example, $\mathrm{Tr}\,(\phi_1\phi_2\{\phi_1, \phi_2\})$ is Q-closed but $\mathrm{Tr}\,(\phi_1\phi_2[\phi_1, \phi_2])$ is not.

Cohomology classes at $N = \infty$

- Introduce formal variables: anticommuting dz^α and commuting $d\theta_i$

$$d\Psi \equiv dz^{\dot\alpha}\,\partial_{z^{\dot\alpha}}\Psi + d\theta_i\,\partial_{\theta_i}\Psi$$

Supercharge action: $\quad Qd\Psi = [\Psi, d\Psi]$

- Single-trace cohomology classes: expanding $\mathrm{Tr}\left[(d\Psi)^n\right]$

$$\partial^{p_1}_{z^+}\partial^{p_2}_{z^-}\partial^{q_1}_{\theta_1}\partial^{q_2}_{\theta_2}\partial^{q_3}_{\theta_3}\,\mathrm{Tr}\Big[\underbrace{(\partial_{z^+}\Psi)^{k_1}(\partial_{z^-}\Psi)^{k_2}(\partial_{\theta_1}\Psi)^{m_1}(\partial_{\theta_2}\Psi)^{m_2}(\partial_{\theta_3}\Psi)^{m_3}}_{\text{symmetrize}}\Big]\Big|_{z^\alpha=0=\theta_i}$$

$$p_\alpha, m_i \in \mathbb{Z}_{\geq 0}\,, \quad q_i, k_\alpha = 0,1$$

Single-trace cohomology and Gravitons

- At infinite N, all the Q-cohomology classes are given by the product $\mathrm{Tr}\,[(d\Psi)^{n_1}]\cdots\mathrm{Tr}\,[(d\Psi)^{n_K}]$.
- Under the AdS/CFT correspondence, the single traces $\mathrm{Tr}\left[(d\Psi)^n\right]$ or the cyclic cohomology classes are dual to single-graviton states in $\mathrm{AdS}_5 \times \mathrm{S}^5$.
- We verified this by matching the Beti numbers of the single-trace cohomology with the number of single graviton states. [CC-Yin'13]
- The products $\mathrm{Tr}\,[(d\Psi)^{n_1}]\cdots\mathrm{Tr}\,[(d\Psi)^{n_L}]$ are dual to multi-graviton states.
- Since $G_N \sim \frac{1}{N^2} \to 0$, multi-gravitons are products of single-gravitons.

Cohomology at Finite N

- The Hilbert space of operators at finite N can be realized as a quotient:

$$\mathscr{H}_N \cong \mathscr{H}_\infty / I_N$$

$\mathscr{H}_\infty$: space of multi-traces ($N = \infty$ Hilbert space)

$\mathscr{H}_N$: finite N Hilbert space, I_N : space of trace identities

(e.g. $2\mathrm{Tr}\,X^3 = 3\mathrm{Tr}\,X\,\mathrm{Tr}\,X^2 - (\mathrm{Tr}\,X)^3$ for $N = 2$)

- A short exact sequence (SES):

$$0 \to I_N \xrightarrow{i} \mathscr{H}_\infty \xrightarrow{\pi} \mathscr{H}_N \to 0$$

i : inclusion map, π : quotient map that imposes the trace identities

Cohomology at Finite N

- Since the maps π and i commutes with the supercharge Q, we have

$$H^n(I_N) \xrightarrow{i_*} H^n(\mathscr{H}_\infty) \xrightarrow{\pi_*} H^n(\mathscr{H}_N)$$

$$\mathrm{im}\,(i_*) = \ker\,(\pi_*)$$

- Some of the cohomology classes (**monotone classes**) at finite N are given by imposing trace identities on the infinite N cohomology classes

$$H^n(\mathscr{H}_\infty)/\mathrm{im}\, i_* \cong \mathrm{im}\, \pi_* \subset H^n(\mathscr{H}_N)$$

- **Fortuitous classes** are defined by the quotient $H^n(\mathscr{H}_N)/\mathrm{im}\,\pi_*$ [CC-Lin'24]

Comments on the bulk duals

- **Conjecture** [CC-Lin'24]:
  - **Monotone classes** $\leftrightarrow$ **Microstates of horizonless geometries**
  - **Fortuitous classes** $\leftrightarrow$ **Microstates of black holes**
- **Evidence/checks**:
  - (Generalized) LLM geometries $\leftrightarrow$ monotone states in $\mathcal{N} = 4$ SYM [CC-Lin'24]
  - Superstrata geometries $\leftrightarrow$ monotone states in D1-D5 CFTs [CC-Lin-Zhang'25]

BPS black holes

IIB String Theory on $\mathrm{AdS}_5 \times \mathrm{S}^5 \longleftrightarrow$ 4d $\mathrm{SU}(N)$ $\mathcal{N} = 4$ SYM

- Supersymmetric black holes in $\mathrm{AdS}_5 \times \mathrm{S}^5$ preserving two supercharges [Gutowski-Reall'04, Chong-Cvetic-Lu-Pope'05, Kunduri-Lucietti-Reall'06]

$\longleftrightarrow$ **Minimally-supersymmetric (1/16 BPS) states in $\mathcal{N} = 4$ SYM**

- These black holes carry five angular momenta:

Rotations in AdS_5: J_1 and J_2 . Rotations in S^5: q_1, q_2, and q_3

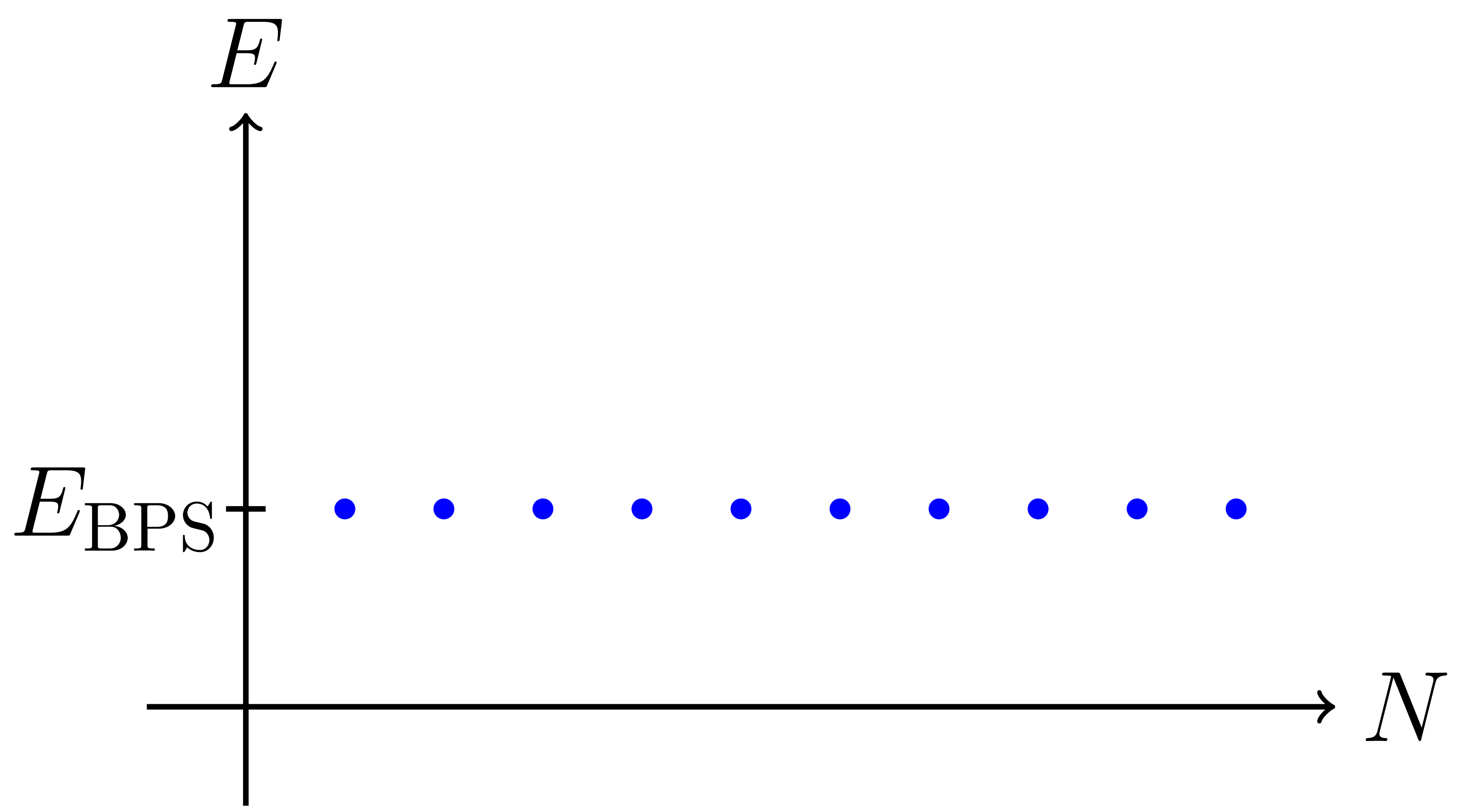
- $J_i, q_a \sim N^2$ and the entropy $S \sim N^2$

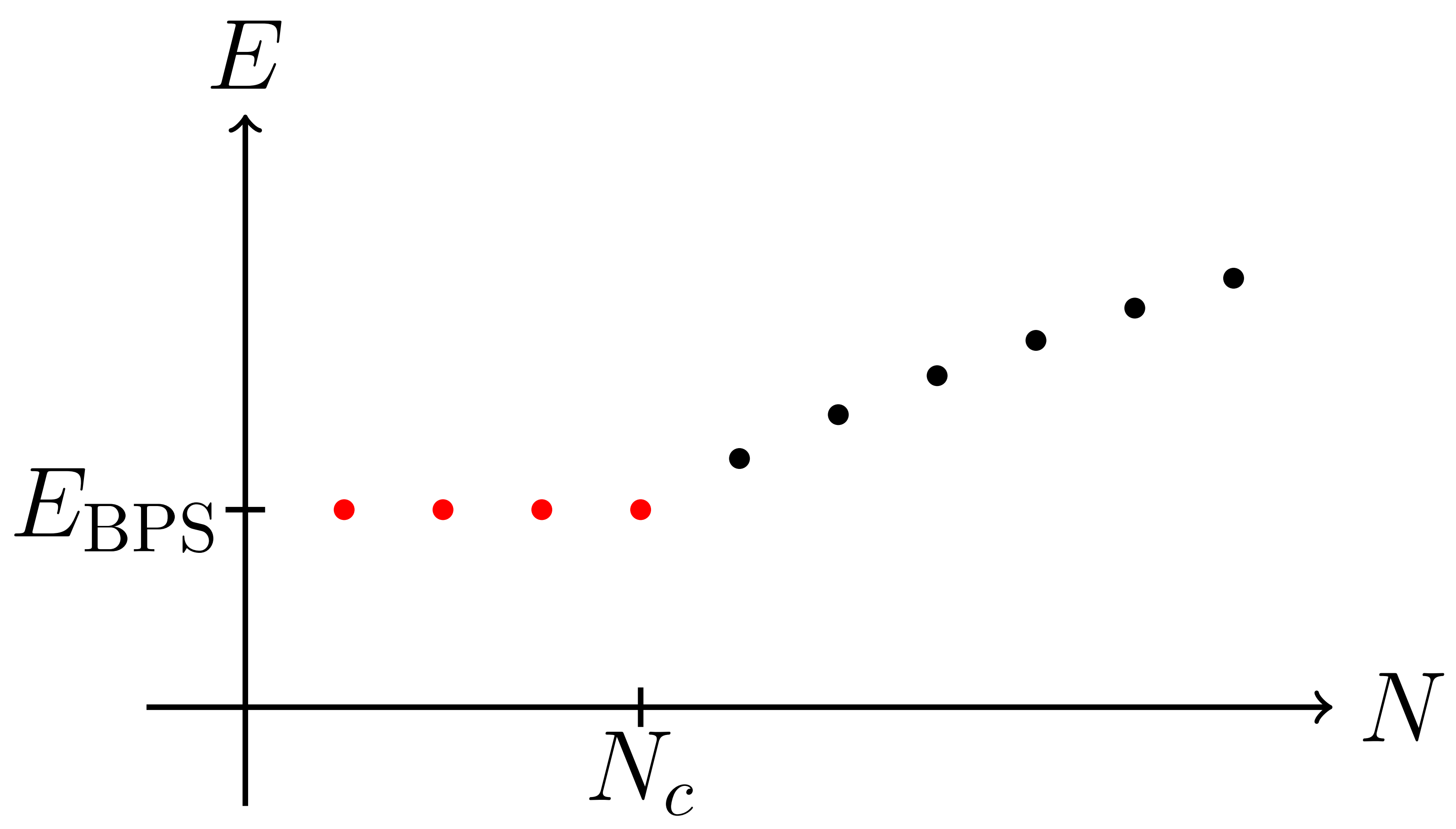
A very intuitive argument:

- Horizonless geometries have a "smooth" $G_N \to 0$ limit — They can be viewed as coherent states of gravitons, and disassemble into non-interacting gravitons as $G_N \to 0$.

- It is highly unlikely for a BPS horizonless solution to become non-BPS as $G_N \to 0$.

- Black hole geometries usually become singular when $G_N \to 0$ (without increasing their energy). For example, BPS BHs in AdS_5.





Tests of the non-renormalization conjectures

$1/8$-BPS Schur sector

- The $1/8$-BPS Schur sector can be defined by a $(Q+S)$-cohomology and is described by a 2d super-chiral algebra. [Beem-Lemos-Liendo-Peelaers-Rastelli-van Rees'13]

- The $(Q+S)$-cohomology is isomorphic to the Q-cohomology with constraints $\partial_{\theta_3}\Psi = 0$ and $\partial_{z_2}\Psi = 0$. [CC-Lin-Wu'23]

- Due to the rigidity of the chiral algebra. We can argue that the strong conjecture is true in this case.

- Generalization: In $\mathcal{N}=2$ SCFT, the $(Q+S)$-cohomology is tree-level (classically) exact for exactly marginal couplings.

$1/8$-BPS chiral ring sector

- The chiral ring is generated by the $\mathcal{N} = 1$ chiral superfields Φ_i and W_α, whose bottom components are $\phi_i = \partial_{\theta_i}\Psi$ and $\lambda_\alpha = \partial_{z^\alpha}\Psi$.

- The superpotential gives the chiral relations: $[\phi_i, \phi_j] = [\phi_i, \lambda_\alpha] = [\lambda_\alpha, \lambda_\beta] = 0$.

- The chiral ring sector is a subsector of the monotone sector given by products of the single traces

$$\mathrm{Tr}\Big[\underbrace{(\partial_{z^+}\Psi)^{k_1}(\partial_{z^-}\Psi)^{k_2}(\partial_{\theta_1}\Psi)^{m_1}(\partial_{\theta_2}\Psi)^{m_2}(\partial_{\theta_3}\Psi)^{m_3}}_{\text{symmetrize}}\Big]\Big|_{z^\alpha=0=\theta_i}$$

- Does the chiral ring receive quantum corrections?

S-duality test

- The $\mathcal{N} = 4$ SYM theory enjoys the S-duality, which maps the theory with gauge group G and complexified gauge coupling

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{\mathrm{YM}}^2}$$

  to the theory with gauge group ${}^L G$ (the Langlands dual of G) and complexified gauge coupling $-1/\tau$.

- Since the dual pair admits weak coupling descriptions near two different points on the space of couplings, the S-duality provides a powerful tool for testing the non-renormalization conjecture.

S-duality test

- The dual pair with gauge groups $\mathrm{SU}(N)$ and $\mathrm{PSU}(N)$ does not give any non-trivial checks because the Q-cohomology depends only on the Lie algebra of the gauge group.

- To perform nontrivial checks, we consider the gauge groups $\mathrm{SO}(2N+1)$ and $\mathrm{USp}(2N)$.

Matching on the Coulomb branch

- Let us move on to a generic point on the Coulomb branch. The gauge group is broken to its maximal torus $\mathrm{U}(1)^N$.

- In convenient basses, for the Cartan-valued superfield Ψ takes the block off-diagonal forms:

$$\Psi_{\mathrm{SO}} = \begin{pmatrix} 0 & i\Psi_D & 0 \\ -i\Psi_D & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Psi_{\mathrm{USp}} = \begin{pmatrix} \Psi_D & 0 \\ 0 & -\Psi_D \end{pmatrix}.$$

Matching on the Coulomb branch

- After removing the zero column and row in Ψ_{SO}, the matrices Ψ_{SO} and Ψ_{USp} are related by a conjugation. Hence, the cohomologies must agree.

- The Coulomb-branch cohomology embeds into the monotone cohomology.

- This is because the fortuitous cohomology is identified with a subspace of the cohomology of trace relations, and all trace relations vanish identically on the Coulomb branch.

Search for non-Coulomb branch classes

- We focus on the simplest dual pair: $\mathrm{SO}(7)$ and $\mathrm{USp}(6)$.

- Focusing on the BMN sector ($\partial_{z^\alpha}\Psi = 0$), Gadde-Lee-Raj-Tomar found the first fortuitous class in the $\mathrm{SO}(7)$ theory with charges $(J_1, J_2, q_1, q_2, q_3) = (\frac{1}{2}, \frac{1}{2}, \frac{5}{2}, \frac{5}{2}, \frac{5}{2})$, which has no S-dual in the $\mathrm{USp}(6)$ theory in the BMN sector.

- If the strong conjecture is true, then there must exist a non-Coulomb branch class with the same charges in the $\mathrm{USp}(6)$ theory outside the BMN sector.

- We did an exhaustive search and did not find any such a class.

Violation of S-duality

- We constructed the cohomology classes up to $L = 3J_1 + 3J_2 + 2q_1 + 2q_2 + 2q_3 = 18$, and found that the $\mathrm{SO}(7)$and $\mathrm{USp}(6)$ cohomology classes agree, except

BMN → $(0,0,3,3,3,8)$; chiral ring → $(1,1,2,2,2,8)$

$(J_1, J_2, q_1, q_2, q_3)$	$(\frac{1}{2}, \frac{1}{2}, \frac{5}{2}, \frac{5}{2}, \frac{5}{2})$		(0,0,3,3,3)	
$(n_+, n_-, n_1, n_2, n_3, n)$	(0,0,3,3,3,8)		(1,1,2,2,2,8)	
gauge group	SO(7)	USp(6)	SO(7)	USp(6)
all states	903	903	826	825
non-Q-closed states	220	221	0	0
Q-exact states	559	559	741	741
monotone classes	123	123	85	84
fortuitous classes	1	0	0	0

Conclusion

- The strong conjecture is false in the $\mathcal{N} = 4$ SYM, i.e., the Q-cohomology must receive loops or non-perturbative corrections.

- The weak conjecture can still be correct, i.e., after taking into account the perturbative and non-perturbative corrections near the free points, the Q-cohomology is independent of the coupling.

- Very surprisingly, the fortuitous class in the BMN sector should be paired with a monotone class in the chiral ring sector.

Conclusion

- Such a chiral ring element should vanish when going onto the Coulomb branch, because the Coulomb-branch cohomology respects the S-duality.
- Naively, this sounds like a contradiction. Since the chiral ring elements are mutually commuting, they should reside in the Cartan subalgebra.
- However, $SO(7)$ admits a commuting triple not simultaneously conjugate into the Cartan subalgebra. [Borel-Freedman-Morgan'1999]
- In general, non-Cartan commuting N-tuple exists in B_N and D_{N+1}.

Open problems

- For B_N and D_N
  - What's the mechanism that lifts the pair of states? How many states are lifted?
  - The perturbative correction to the Q-cohomology can be computed in the holomorphic twisted theory. [Budzik-Gaiotto-Klup-Williams-Wu-Yu'23]
  - One can argue that the perturbative corrections truncate at finite loop orders. It would be very interesting to work out the explicit computation.

Protection in the chiral-ring sector?

- For A_N and C_N,
  - There is no non-Cartan commuting N-tuple. This implies that the chiral ring is in the Coulomb-branch cohomology.
  - The Coulomb branch has no quantum correction at the level of the two-derivative action.
  - Is the strong conjecture true for the chiral-ring sector?
- There are still many things to be understood!

Thank you