Bootstrapping Thermal CFTs

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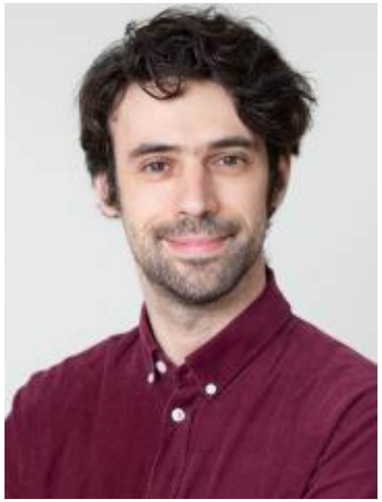


Julien Barrat

[2306.12417 & 2312.13030 Marchetto, Miscioscia, EP]

[2407.14600 Barrat, Fiol, Marchetto, Miscioscia, EP]

[2411.00978 Barrat, Marchetto, Miscioscia, EP]

[2506.06422 Barrat, Bozkurt, Marchetto, Miscioscia, EP]

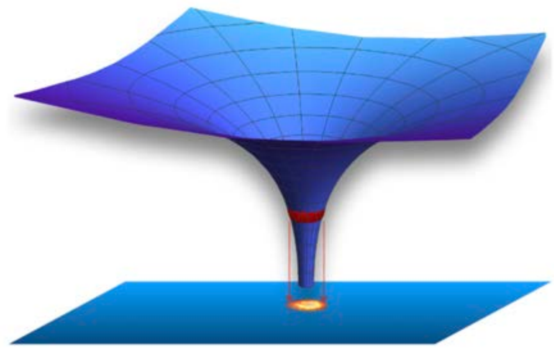
[2510.20894 Barrat, Bozkurt, Marchetto, Miscioscia, EP]

Motivation

* Quantum critical points: nonzero temperature in the lab.

* Black Holes through AdS/CFT.

* CFTs on non-trivial manifolds.



Thermal CFTs

Thermal effects are captured by placing the theory on a **circle**

$$S^1_\beta \times \mathbb{R}^{d-1} \qquad \beta = \frac{1}{T}$$

With periodic boundary conditions for the bosons and anti-periodic for the fermions.

Translations	✓
Spatial rotations	✓
Boosts	✗
Dilatations	✗
Special conformal	✗

Broken symmetries are not lost but captured by broken Ward Identities.

Thermal CFTs

Assume we know the **zero temperature CFT data**:

$$\Delta \ , \ f_{123}$$

and are interested in computing **new finite temperature data:**
the non-zero ***thermal one-point functions***:

$$\langle \mathcal{O}(x) \rangle_\beta = \frac{b_{\mathcal{O}}}{\beta^{\Delta_{\mathcal{O}}}}$$

for neutral scalar operators.

And more generally for all traceless symmetric tensors:

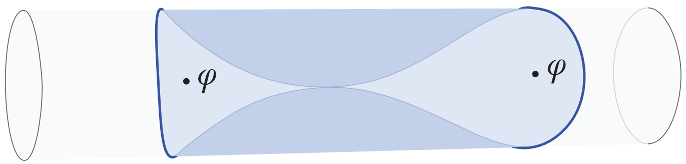
$$\langle \mathcal{O}^{\mu_1 \cdots \mu_J}(x) \rangle_\beta = \frac{b_{\mathcal{O}_J}}{\beta^{\Delta_{\mathcal{O}_J}}} (e^{\mu_1} \cdots e^{\mu_J} - traces)$$

Thermal CFTs

We can still use the OPE:

$$\mathcal{O}_1(x) \times \mathcal{O}_2(0) = \sum_{\mathcal{O}} f_{12\mathcal{O}} |x|^{\Delta_{\mathcal{O}}-\Delta_1-\Delta_2-J} x_{\mu_1} \ldots x_{\mu_J} \mathcal{O}^{\mu_1 \ldots \mu_J}(0)$$

But now the radius of convergence is finite: $|x| < \beta$



$$|x| = \sqrt{x^2 + \tau^2}$$

$$x^2 = ||\vec{x}||^2$$

[Iliesiu, Koloğlu, Mahajan, Perlmutter, Simmons-Duffin 2018]

Thermal 2pt functions

The **two-point function** of identical scalars ϕ, using the OPE

$$\langle \phi(\tau, r)\phi(0)\rangle_\beta = \sum_{\mathcal{O}} f_{\phi\phi\mathcal{O}} |x|^{\Delta_{\mathcal{O}}-2\Delta_\phi-J} x_{\mu_1}\ldots x_{\mu_J} \langle \mathcal{O}^{\mu_1\cdots\mu_J}\rangle_\beta$$

$$\langle \mathcal{O}^{\mu_1\cdots\mu_J}(x)\rangle_\beta = \frac{b_{\mathcal{O}_J}}{\beta^{\Delta_{\mathcal{O}_J}}}(e^{\mu_1}\cdots e^{\mu_J} - tr)$$

and the definition of the Gegenbauer polynomials:

$$\langle \phi(\tau, r)\phi(0)\rangle_\beta = \sum_{\mathcal{O}} \frac{a_{\mathcal{O}}}{\beta^{\Delta_{\mathcal{O}}}} |x|^{\Delta_{\mathcal{O}}-2\Delta_\phi} C_J^{(\nu)}\left(\frac{\tau}{|x|}\right)$$

$$a_{\mathcal{O}} = b_{\mathcal{O}} f_{\mathcal{O}\phi\phi} \frac{J!}{2^J(\nu)_J} \qquad (\nu)_J = \frac{\Gamma(\nu+J)}{\Gamma(\nu)} \qquad \nu = \frac{d-2}{2} \qquad |x| = \sqrt{x^2+\tau^2}$$

New Finite Temperature data

Thermal crossing

Periodicity of the two-point function is captured by the **KMS condition**: [Kubo 1957] [Martin,Schwinger 1959]

$$\langle \phi(\tau, x)\phi(0,0)\rangle_\beta = \langle \phi(\tau+\beta, x)\phi(0,0)\rangle_\beta$$

The OPE expression does not manifestly satisfy KMS, thus imposing it gives a novel nontrivial **consistency condition**.

Variation of KMS:

$$\left\langle \phi\left(\beta/2+\tau\right)\phi(0)\right\rangle_\beta = \left\langle \phi\left(\beta/2-\tau\right)\phi(0)\right\rangle_\beta$$

[El-Showk, Papadodimas 2011]

Plan of the talk

- ✱ **Sum rules from KMS** and lesson for light operators
- ✱ Heavy operators: **asymptotic OPE density**
- ✱ A **Numerical** approach: Light + Heavy
- ✱ An **Analytical** approach: Dispersion Relation
- ✱ Application to **Holographic CFTs**

Sum Rules

[2312.13030 Marchetto, Miscioscia, EP]

Sum rules from KMS

* OPE expand both sides of the El-Showk - Papadodimas formula

$$\left\langle \phi\left(\beta/2+\tau,x\right)\phi(0)\right\rangle_\beta = \left\langle \phi\left(\beta/2-\tau,x\right)\phi(0)\right\rangle_\beta$$

* Then further expand the result in powers of τ and x.
* Use the definition of Gegenbauer polynomials and the binomial theorem.

$$\sum_{\mathcal{O}\in\phi\times\phi} b_{\mathcal{O}}\, f_{\mathcal{O}\phi\phi}\, F_{\ell,n}(h,J) = 0$$

$$\ell \in 2\mathbb{N}+1\ ,\ n\in\mathbb{N}$$

$$h = \Delta - J$$

$$F_{\ell,n}(h,J) = \frac{1}{2^{h+J}}\binom{\frac{h-2\Delta_\phi}{2}}{n}\binom{h+J-2\Delta_\phi-2n}{\ell}\ {}_3F_2\left[\begin{matrix}\frac{1-J}{2}, -\frac{J}{2}, \frac{h}{2}-\Delta_\phi+1 \\ \frac{h}{2}-\Delta_\phi-n+1, -J-\nu+1\end{matrix}\middle| 1\right]$$

Sum rules from KMS

$$\sum_{\mathcal{O}\in\phi\times\phi} \underline{b_{\mathcal{O}}}\, \underline{f_{\mathcal{O}\phi\phi} F_{\ell,n}}(h,J) = 0$$

$\ell \in 2\mathbb{N}+1\ ,\ n \in \mathbb{N}$

$h = \Delta - J$

New

Finite Temperature

Known data

Zero Temperature

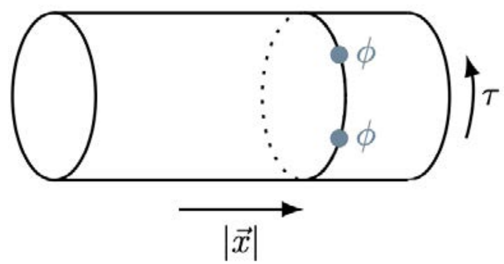
We have an **infinite set** of **linear equations** for the combinations $a_{\mathcal{O}} \propto b_{\mathcal{O}} f_{\mathcal{O}\phi\phi}$

Difficulty: **Thermal one-point functions *not*** sign-definite.

Which is crucial for linear programming methods (standard numerical bootstrap).

Sum rules from KMS

The sum rules for $a_{\mathcal{O}} \propto b_{\mathcal{O}} f_{\mathcal{O}\phi\phi}$



further **simplified for** $x = 0$

$$\frac{\Gamma\left(2\Delta_\phi + \ell\right)}{\Gamma\left(2\Delta_\phi\right)} = \sum_{\Delta \neq 0} \frac{a_\Delta}{2^\Delta} \frac{\Gamma\left(\Delta - 2\Delta_\phi + 1\right)}{\Gamma\left(\Delta - 2\Delta_\phi - \ell + 1\right)} \qquad \ell \in 2\mathbb{N} + 1$$

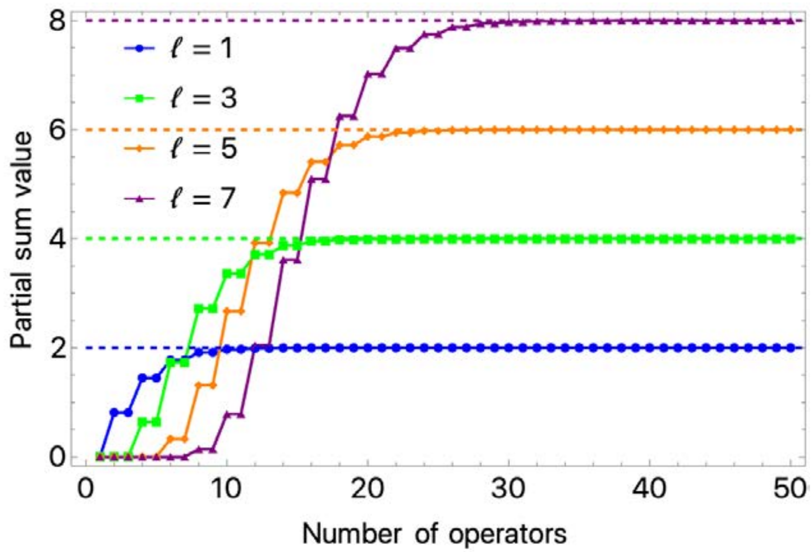
$a_\Delta = \sum_{\mathcal{O} \in \phi \times \phi} a_{\mathcal{O}}\, C_J^{(\nu)}(1)$ for fixed Δ

Operators of same Δ but different J cannot be distinguished because of $x = 0$.

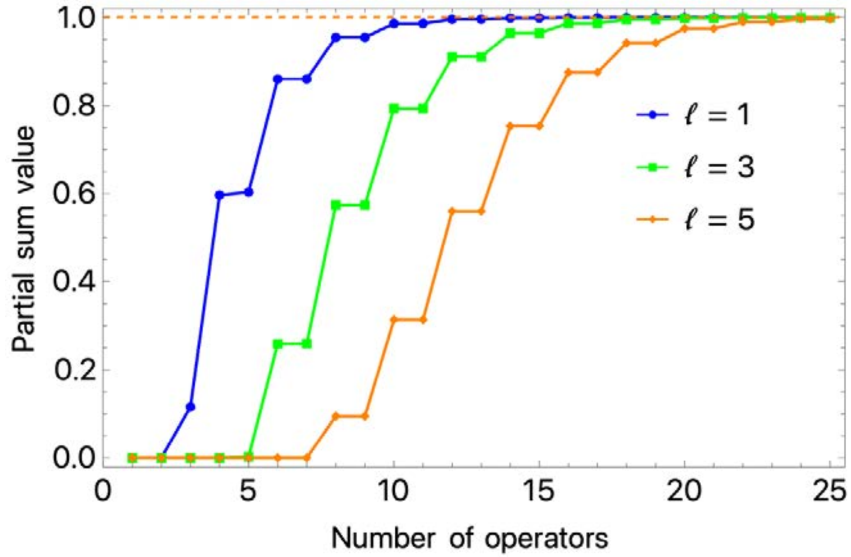
Generically this does not occur, unless there is extra symmetry, like for free theory.

KMS sum rules: test & learn

4-dim free theory



$\mathrm{O}(N)$ **model at large N**



Dashed straight lines: the LHS of the sum rule (identity contribution). The RHS plot adding operators.

$$\frac{\Gamma\left(2\Delta_\phi + \ell\right)}{\Gamma\left(2\Delta_\phi\right)} = \sum_{\Delta \neq 0} \frac{a_\Delta}{2^\Delta} \frac{\Gamma\left(\Delta - 2\Delta_\phi + 1\right)}{\Gamma\left(\Delta - 2\Delta_\phi - \ell + 1\right)}$$

Observation: for **small** ℓ only **few light operators** contribute.

Heavy Operators

[2312.13030 Marchetto, Miscioscia, EP]

Asymptotic thermal OPE density

Consider the simplified two-point function at $x = 0$:

$$\langle \phi(\tau)\phi(0) \rangle_\beta = \tau^{-2\Delta_\phi} \sum_{\mathcal{O} \in \phi \times \phi} a_{\mathcal{O}} \frac{\tau^{\Delta_{\mathcal{O}}}}{\beta^{\Delta_{\mathcal{O}}}} = \tau^{-2\Delta_\phi} \int_0^\infty d\Delta\, \rho(\Delta)\, \frac{\tau^{\Delta}}{\beta^{\Delta}}$$

via introducing the spectral density $\rho(\Delta) = \sum_{\Delta'} \delta(\Delta' - \Delta)\, a_\Delta$

Asymptotic thermal OPE density

Using Tauberian theorems or Laplace transform the contribution of the identity:

$$\rho(\Delta) \overset{\Delta\to\infty}{\sim} \frac{1}{\Gamma(2\Delta_\phi)}\Delta^{2\Delta_\phi-1}$$

Keep in mind, the physical spectrum is discrete $\rho(\Delta) \overset{\Delta\to\infty}{\sim} \sum_{\Delta'} \delta(\Delta'-\Delta)\, a_\Delta$

More correctly: **average density** of OPE **of Heavy operators**

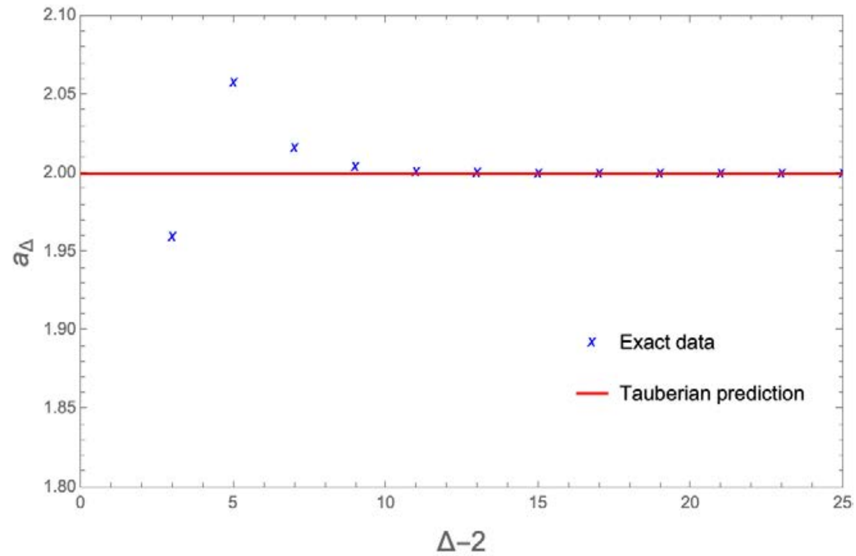
$$\int_0^\Delta \rho(\tilde\Delta)d\tilde\Delta \overset{\Delta\to\infty}{\sim} \frac{\Delta^{2\Delta_\phi}}{\Gamma(2\Delta_\phi+1)}\left(1+\mathcal{O}\left(\frac{1}{\Delta}\right)\right)$$

Ex. 3d O(N) at large N

$$\langle\phi_i(\tau,x)\phi_j(0,0)\rangle_\beta = \delta_{ij}\sum_{m=-\infty}^{\infty}\int\frac{d^2k}{(2\pi)^2}\frac{e^{-i\vec{k}\cdot\vec{x}-i\omega_m\tau}}{\omega_n^2+\vec{k}^2+m_{th}^2} = \delta_{ij}\sum_{m=-\infty}^{\infty}\frac{e^{-m_{th}\sqrt{(\tau+m\beta)^2+x^2}}}{\sqrt{(\tau+m\beta)^2+x^2}}$$

[Sachdev, Ye 1992]

$$a_\Delta \overset{\Delta\to\infty}{\simeq} 2$$

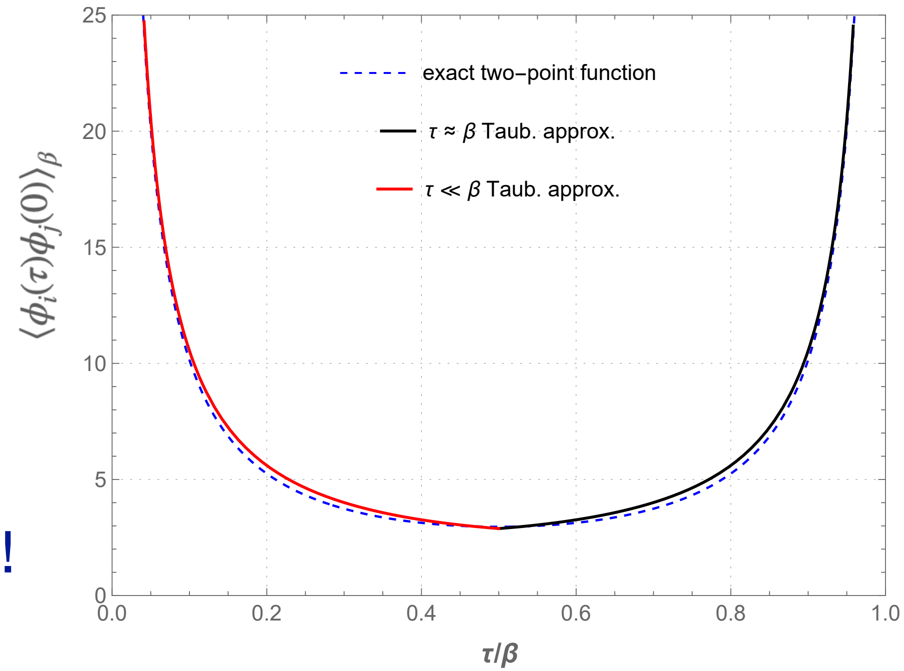


$$\Delta_{\phi_i} = \frac{1}{2}+\mathcal{O}\left(\frac{1}{N}\right) \qquad a_\Delta \overset{\Delta\to\infty}{\simeq} \frac{\Delta^{2\Delta_\phi-1}}{\Gamma(2\Delta_\phi)}\delta\Delta$$

$$\Delta_\sigma = 2+\mathcal{O}\left(\frac{1}{N}\right)$$

$$\langle\phi(\tau)\phi(0)\rangle_\beta \simeq \begin{cases}\int_0^\infty d\Delta\frac{\Delta^{2\Delta_\phi-1}}{\Gamma(2\Delta_\phi)}\frac{(\beta-\tau)^{\Delta-2\Delta_\phi}}{\beta^\Delta} & \tau/\beta \ll 1\\ \int_0^\infty d\Delta\frac{\Delta^{2\Delta_\phi-1}}{\Gamma(2\Delta_\phi)}\frac{\tau^{\Delta-2\Delta_\phi}}{\beta^\Delta} & \tau/\beta \sim 1\end{cases}$$

Tauberian approximation: the two-point function with **less than 10% error!**



The Tauberian approximation is very good!

Numerical Approach

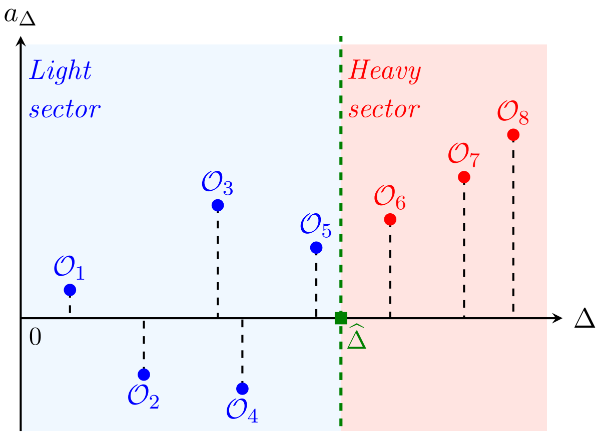
[2411.00978 Barrat, Marchetto, Miscioscia, EP]

Numerical method

Inspired by [Gliozzi 2013] [Poland, Prilepina, Tadić’ 2023] [W. Li 2023]

1. **Input**: zero Temperature spectrum and **Output**: a_Δ & c_i .

2. Truncate the sum + *improved* Tauberian asymptotic:



$$f(\ell) = \sum_{\Delta<\hat{\Delta}} a_\Delta \, F(\Delta,\ell) + \sum_{\Delta>\hat{\Delta}} a_\Delta^T \, F(\Delta,\ell)$$

Light operators

Heavy operators: Tauberian

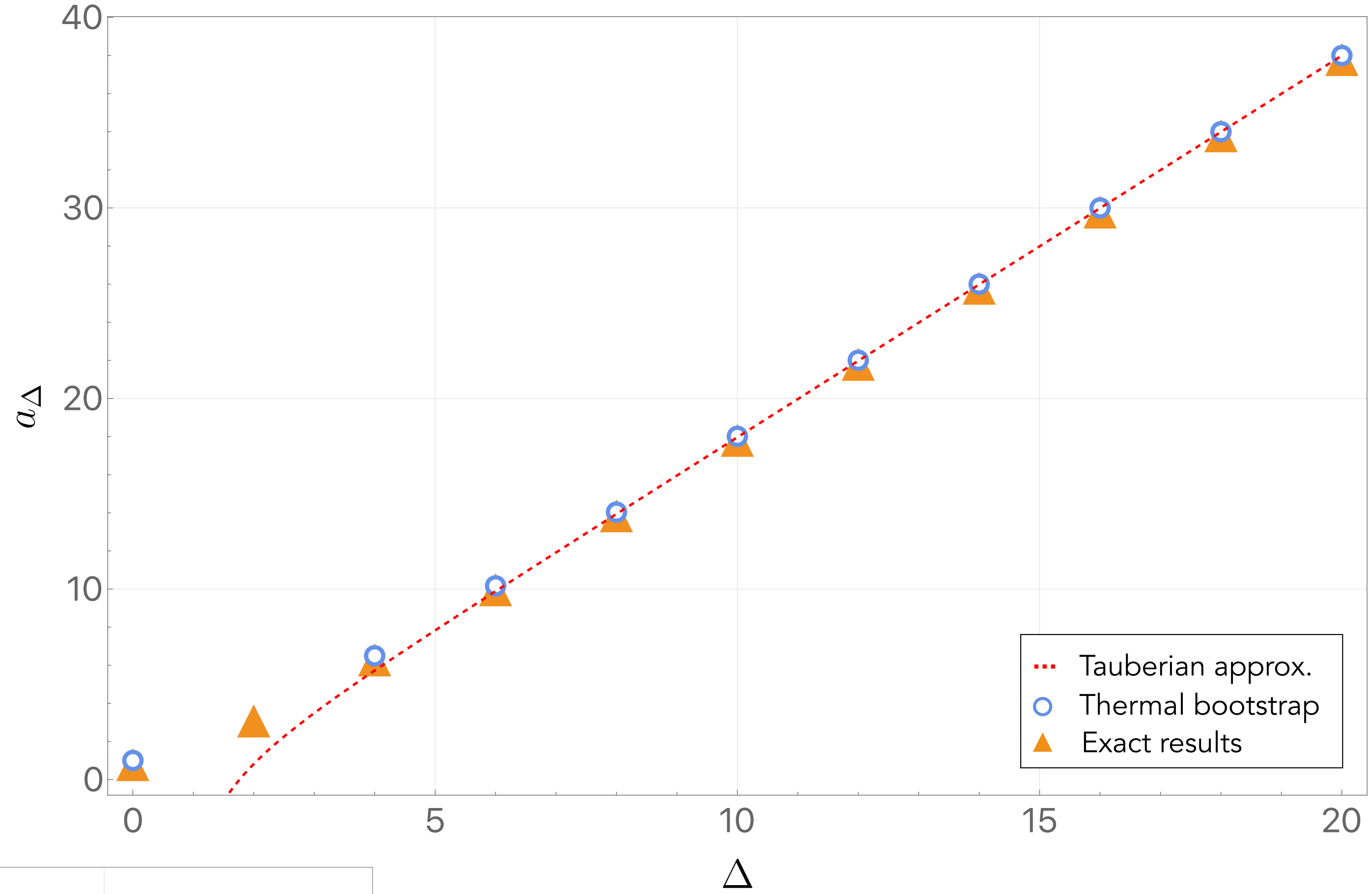
$$a_\Delta^T \sim \frac{\Delta^{2\Delta_\phi - 1}}{\Gamma(2\Delta_\phi)} \delta\Delta \left(1 + \frac{c}{\Delta} + \ldots\right)$$

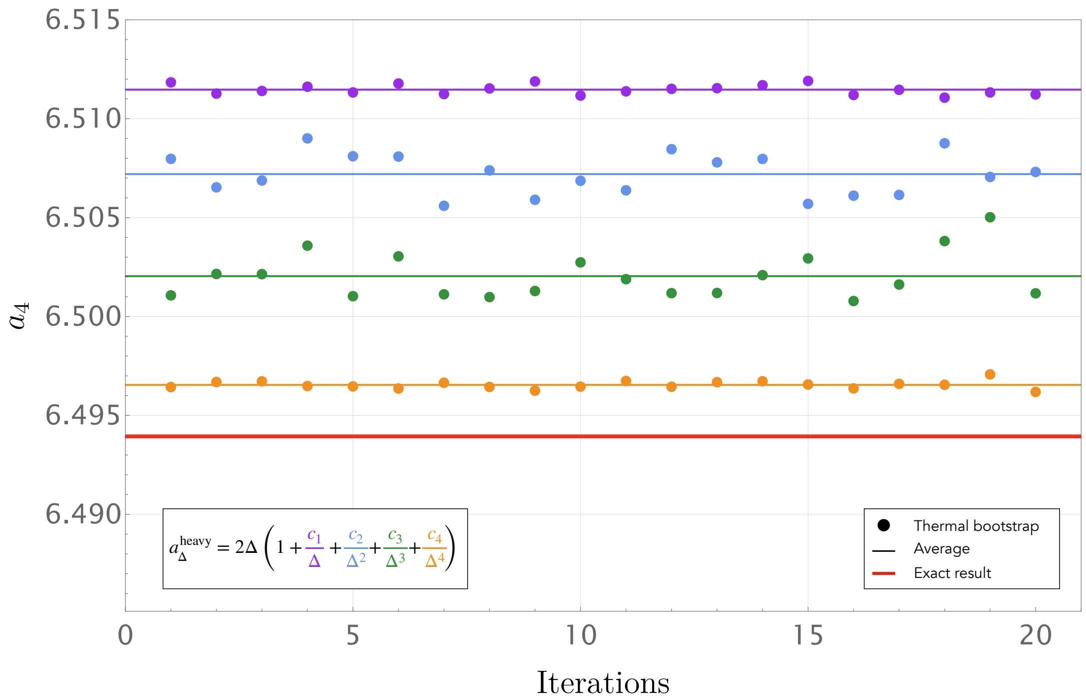
3. Numerically minimize with “random” coefficients the square of the sum rules.

$$\min \left[\sum_{\ell \leq \ell_{max}} r_\ell \, f^2(\ell) \right]$$

Random coefficients

4d Free theory with $\Delta_\phi = 1$

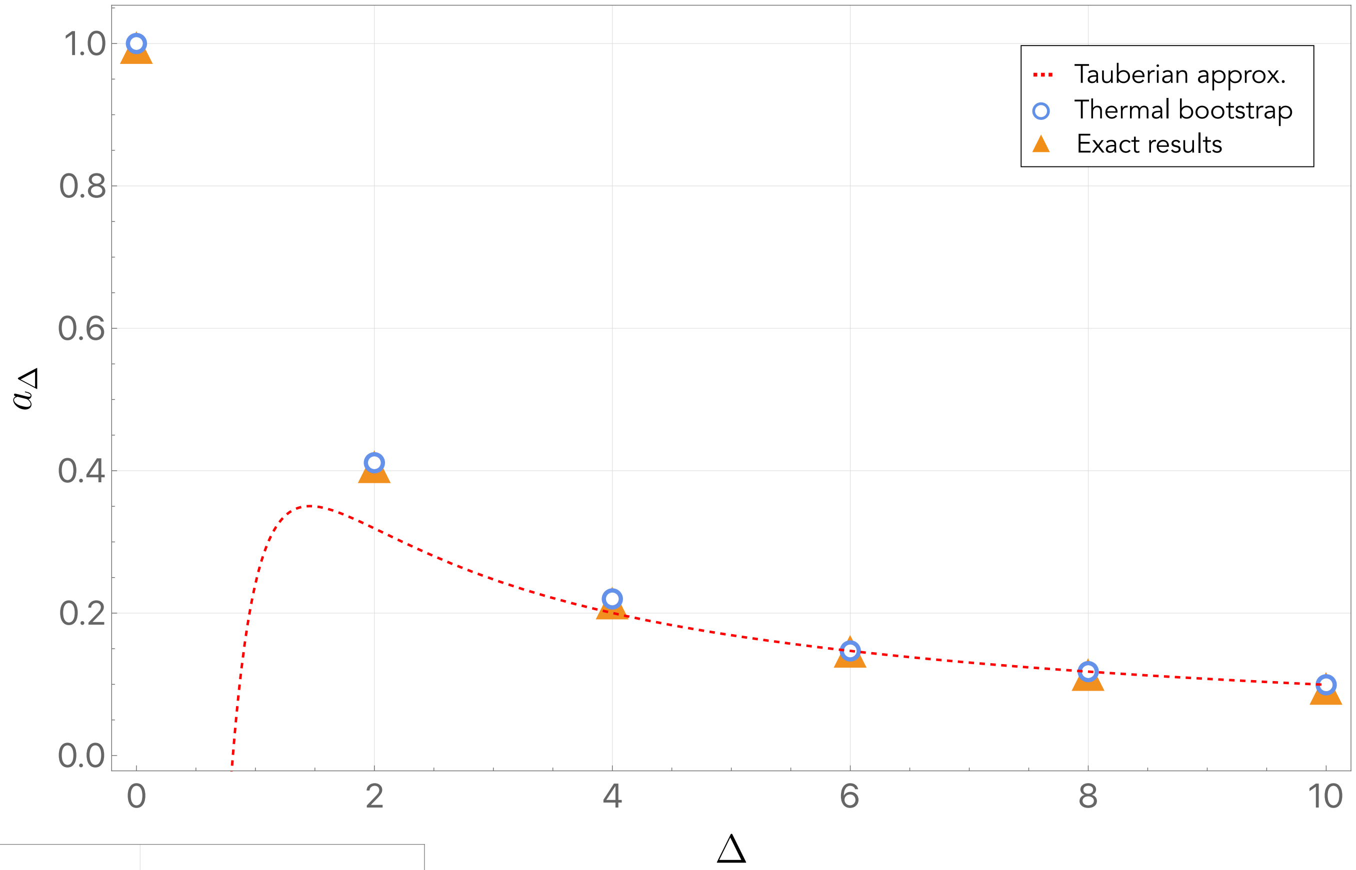


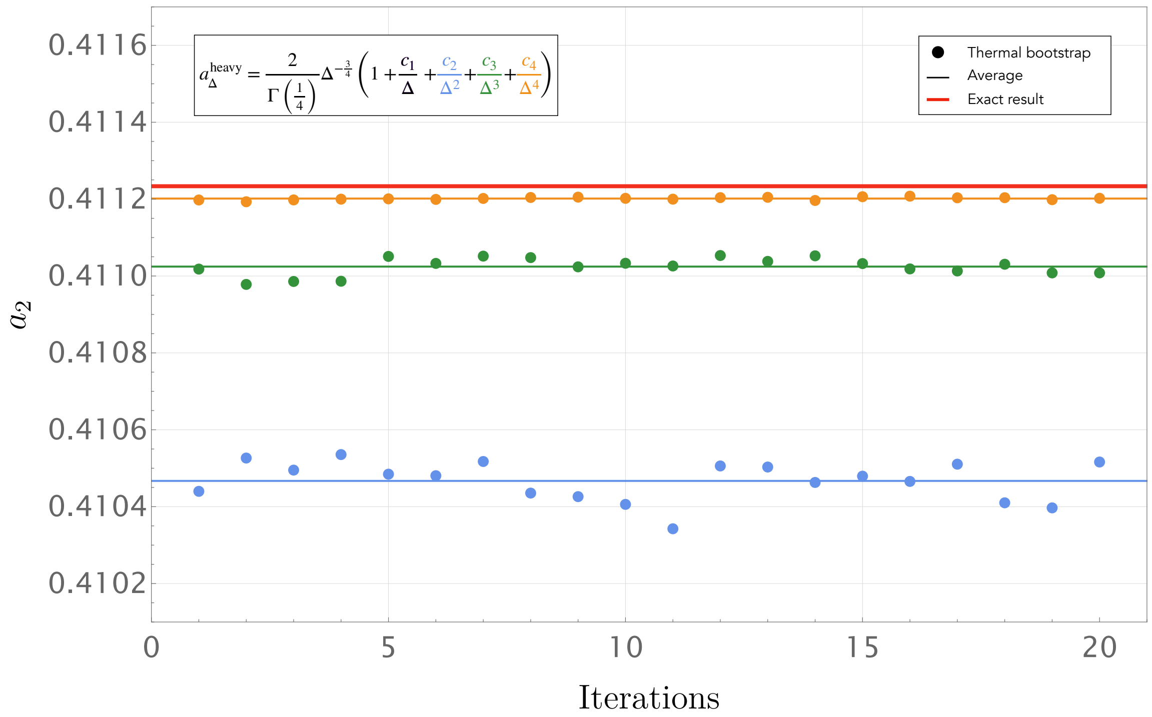


The **systematic error** reduces as we add more **corrections** to the **Tauberian** approximation.

2d Ising

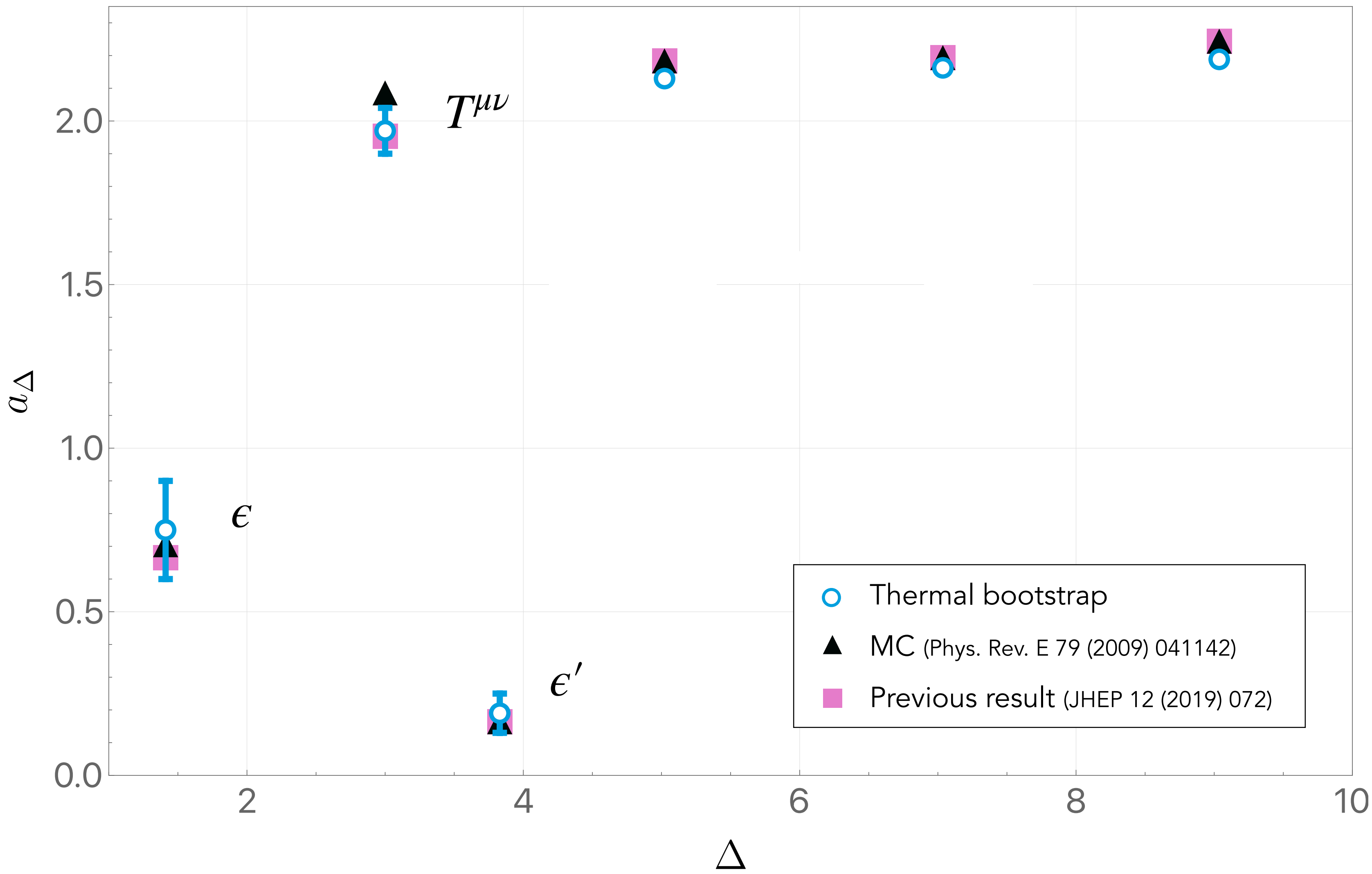
$\langle \sigma\sigma \rangle_\beta$



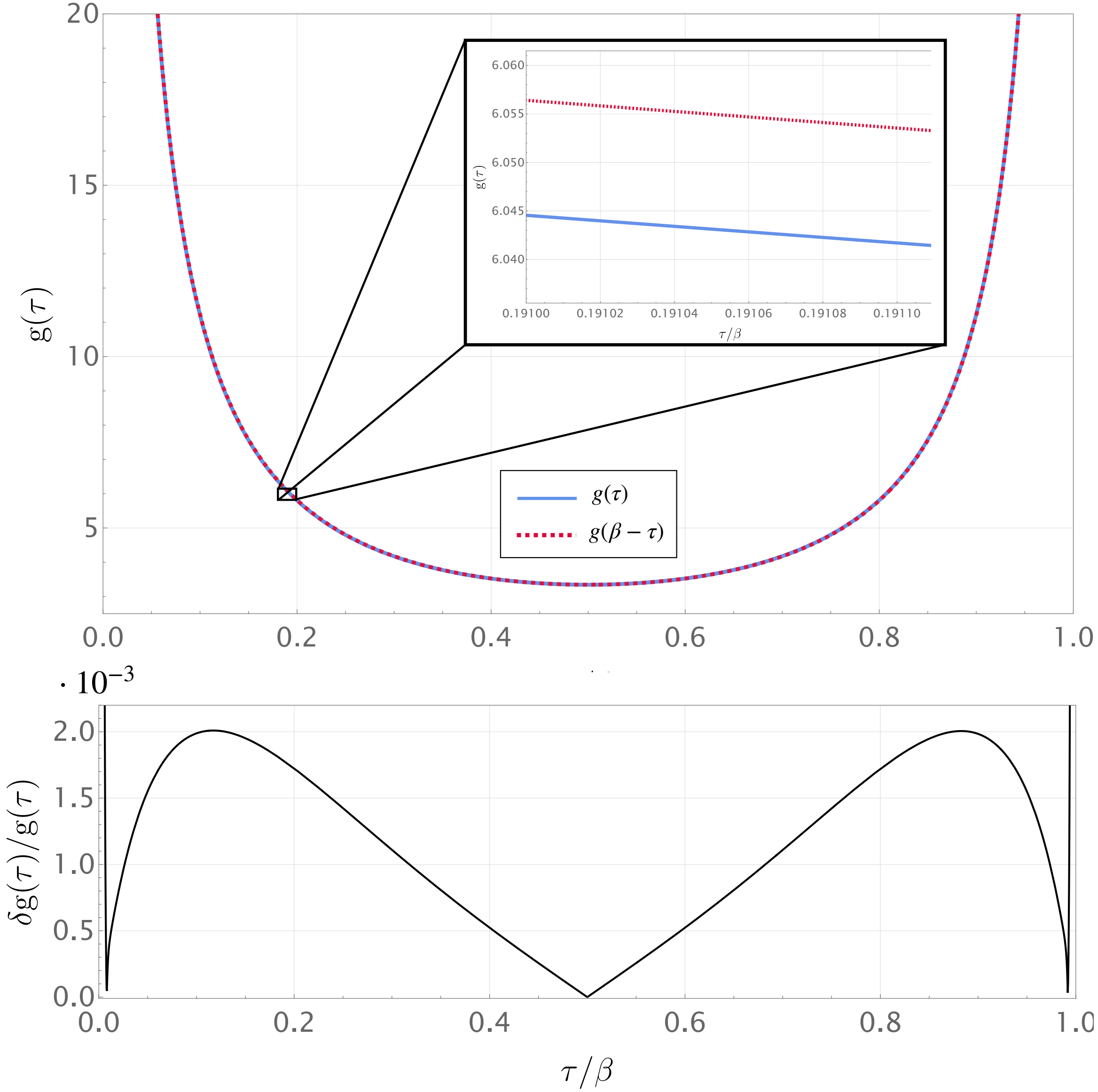


The **systematic error** reduces as we add more **corrections** to the **Tauberian** approximation.

3d Ising $\langle \sigma\sigma \rangle_\beta$



3d Ising $\langle \sigma\sigma \rangle_\beta$



$$g(\tau) = \tau^{-2\Delta_\phi} \sum_\Delta \frac{a_\Delta}{\beta^\Delta} \tau^\Delta$$

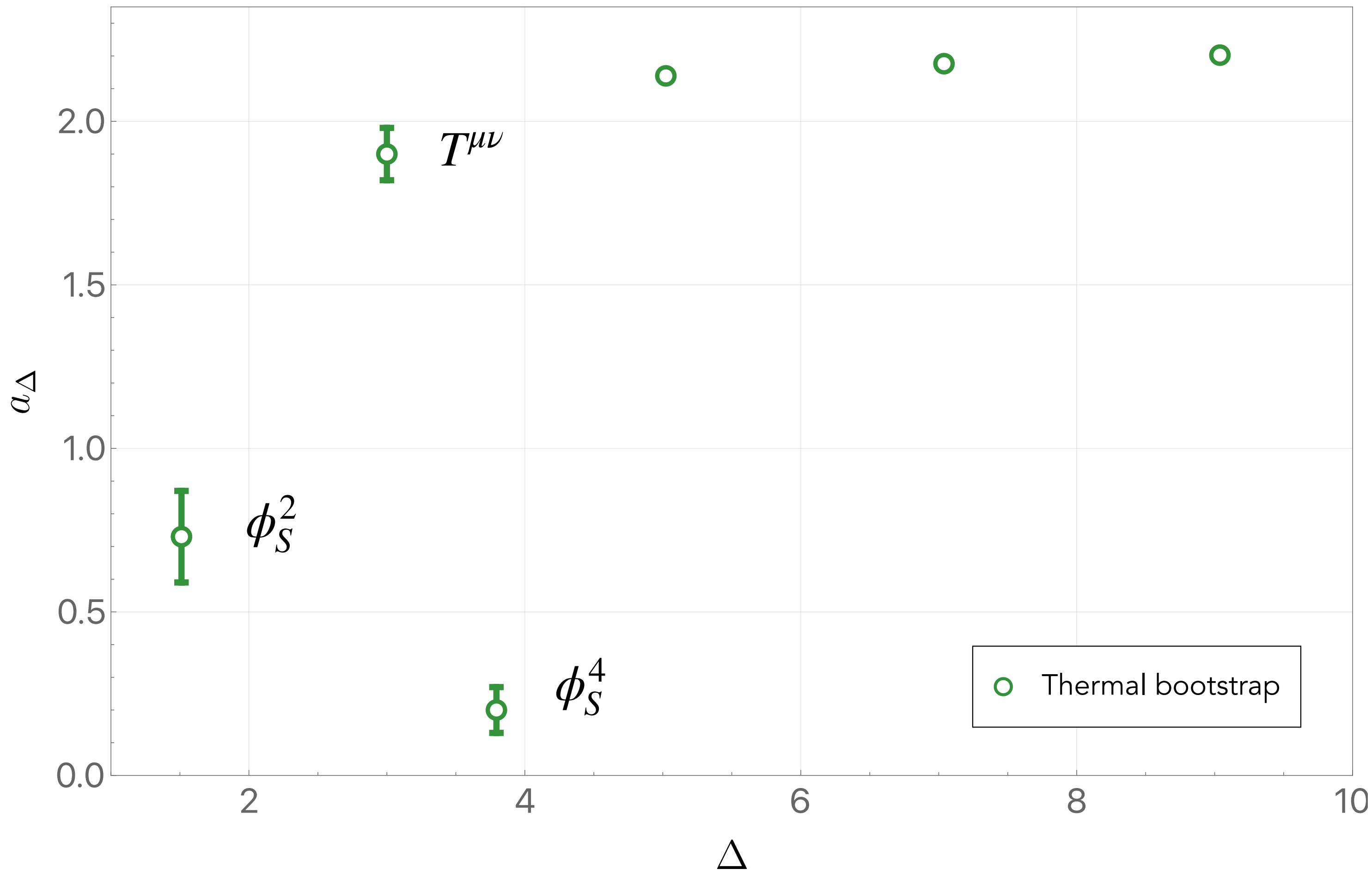
$$\frac{\partial^\ell}{\partial \tau^\ell}\left[g\left(\frac{\beta}{2}+\tau\right) - g\left(\frac{\beta}{2}-\tau\right)\right]_{\tau=0} = 0$$

The failure of KMS is only of order 10^{-3}!

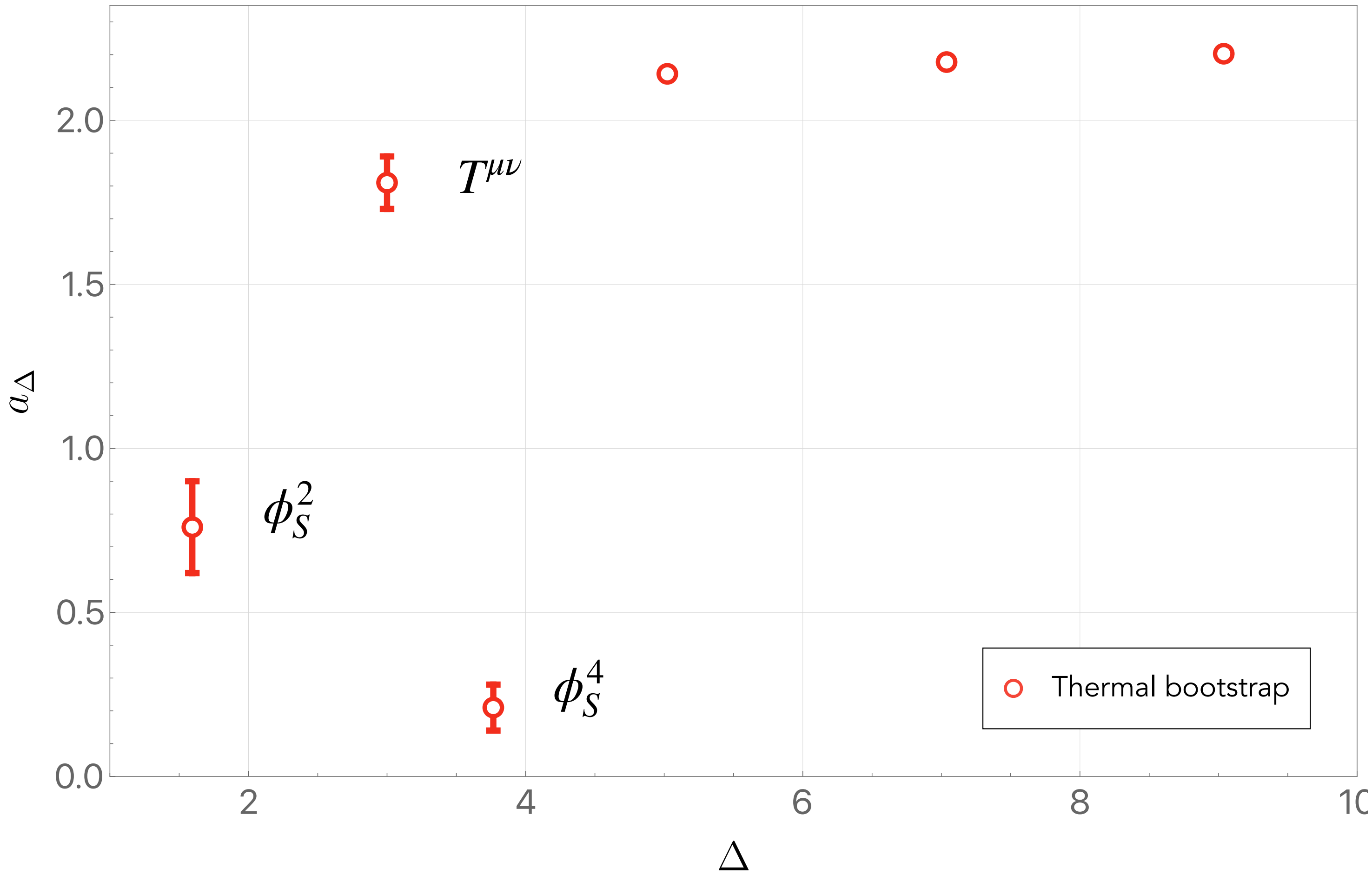
Keeping only the first 3 light operators and seven derivatives

Each plot takes a couple of minutes on a laptop!

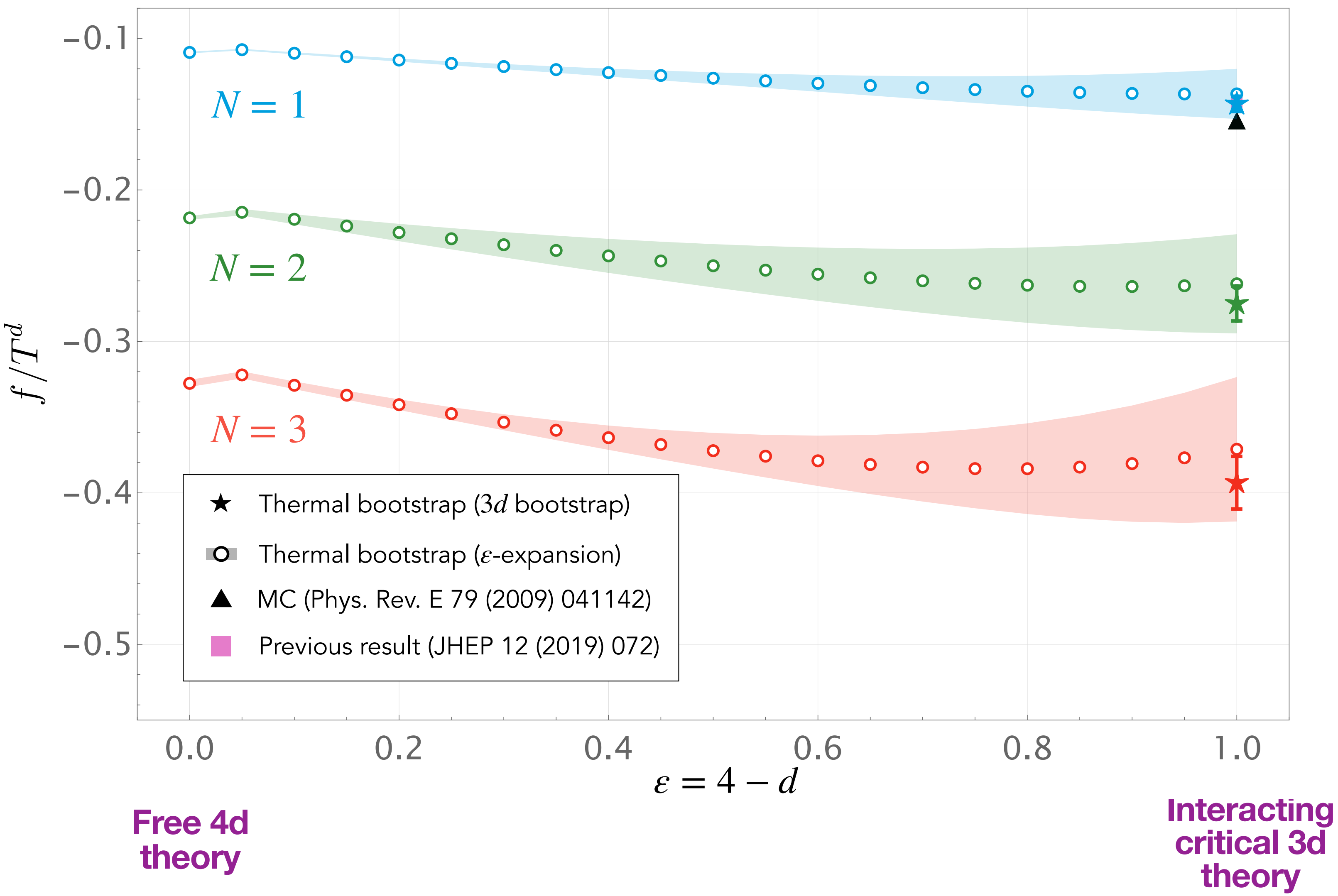
The 3d $O(2)$ XY model



The 3d $O(3)$ Heisenberg model



Free energy across dimensions



[2505.20403 Bulgarelli,Caselle,Nada,Panero] new MC: systematic shift but qualitative trend is the same

Analytic Approach

[2506.06422 Barrat, Bozkurt, Marchetto, Miscioscia, EP]

Analytic bootstrap

Zero Temperature

Large spin expansion

[Fitzpatrick,Kaplan,Poland,Simmons-Duffin 2012]
[Komargodski, Zhiboedov 2012]

Inversion formula

[Caron-Huot 2017]

Dispersion Relation

[Carmi, Caron-Huot 2019]

Thermal

Tauberian approximation

[Marchetto, Miscioscia, EP 2023]

Applied to 3d Ising

[Iliesiu, Koloğlu, Mahajan, Perlmutter, Simmons-Duffin 2018]

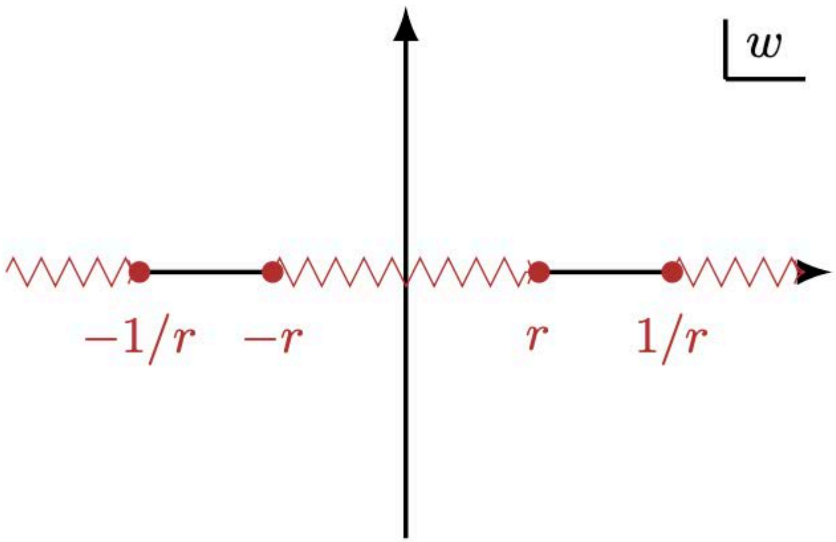
Proposed but never used

[Alday, Koloğlu, Zhiboedov 2020]

Analytic structure

The analytic structure of the thermal 2pt function was already in

[Iliesiu, Koloğlu, Mahajan, Perlmutter, Simmons-Duffin 2018]



$$z = \tau + ix = rw$$

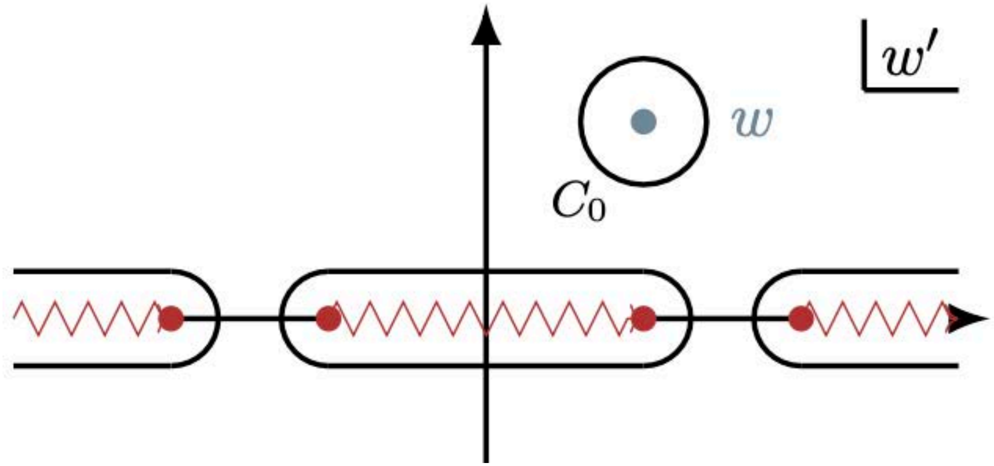
$$\bar{z} = \tau - ix = rw^{-1}$$

We want to learn how to reconstruct the 2pt functions from it.

Dispersion Relation

The thermal dispersion relation was written in:

[Alday, Koloğlu, Zhiboedov 2020]



$$g_{DR}(r,w) = \oint_{C_0} \frac{dw'}{2\pi i} \frac{g(r,w')}{w'-w} = \int_0^r \frac{dw'}{\pi i} \frac{w^2(1-w')(1+w')(1+w'^2)}{w'(w'-w)(w'+w)(1-ww')(1+ww')} \text{Disc}\, g(r,w')$$

$\Rightarrow$ The Kernel $K(w,w')$

But never tested or used. We put it to the use combining it with OPE decomposition!

4d Free theory

We can compute the discontinuity of the two point function

$$g(z, \bar{z}) = \sum_{\mathcal{O}} \frac{a_{\mathcal{O}}}{\beta^{\Delta}} f_{\Delta,J}(z, \bar{z})$$

block by block in the OPE expansion: i.e. Disc$[g(z, \bar{z})] = \sum_{\mathcal{O}} \frac{a_{\mathcal{O}}}{\beta^{\Delta}}$Disc$[f_{\Delta,J}(z, \bar{z})]$

$$f_{\Delta,J}(z, \bar{z}) = \left((1-z)(1-\bar{z})\right)^{\frac{\Delta}{2}-\Delta_\phi} C_J^{(\nu)}\left(\frac{z+\bar{z}}{2\sqrt{z\bar{z}}}\right)$$

For $\Delta_\phi = 1$ and $d = 4$ only the identity operator contributes to the discontinuity.

$$\text{Disc}\,[\mathbf{1}] = \text{Disc}\,\left((1-z)(1-\bar{z})\right)^{-\Delta_\phi} = -2i\sin(\pi\Delta_\phi)\left((1-z)(\bar{z}-1)\right)^{-\Delta_\phi}\Theta(\bar{z}-1)$$

Naive Free theory

Doing the integral we get:

$$g_{DR}(z,\bar{z}) = \int_0^r K(w,w')\,\text{Disc}\,[\mathbf{1}] = \frac{1}{(1-z)^{\Delta_\phi}(1-\bar{z})^{\Delta_\phi}} + \frac{1}{(1+z)^{\Delta_\phi}(1+\bar{z})^{\Delta_\phi}}$$

<u>This result has three problems:</u>

- ✱ It is **not** KMS invariant.
- ✱ After block expansion: **incorrect** $a_{\mathcal{O}}$'s.
- ✱ The identity contribution is **missing**.

$\mathcal{O}$	$a_{\mathcal{O}}$ exact	from DR	from LIF
$\mathbf{1}$	1	0	0
$[\phi\phi]_{0,J}$	$2\zeta(2+J)$	2	2

Correct Free theory

We found two ways of fixing all those problems:

* First attempt: add “arcs” and a “KMS compensator”.

* More elegant: make the DR KMS invariant to start with.

Inspired by [Dalimil Mazáč 2018]

Both give the same result. Just saw the second one today.

KMS Dispersion Relation

Inspired by [Dalimil Mazáč 2018]

Making the DR KMS invariant from the beginning:

$$g_{KMS\,DR}(z,\bar{z}) = \frac{1}{2}\sum_{m\in\mathbb{Z}} g_{DR}(z-m,\bar{z}-m)$$

* This is automatically KMS invariant.
* Directly gives the correct thermal OPE coefficients.
* The identity contribution is included, no arcs needed.

O(N) model in ε-expansion

Now two operators contribute to the discontinuity:

$$\phi \times \phi = 1 + \phi^2$$

This is because the conformal dimension of ϕ^2 is not longer integer

$$\Delta_{\phi^2} = 2 + \epsilon\gamma_{\phi^2}$$

And all the higher operators have anomalous dimensions of order ϵ^2.

So to this order they do not contribute to the discontinuity.

O(N) model in ε-expansion

We have two new contributions at order ϵ:

* One from expanding the identity contribution around $\Delta_\phi = 1 - \dfrac{\epsilon}{2}$

$$g_{DR}(z, \bar{z}) = \int_0^r K(w, w')\,\text{Disc}\,[\mathbf{1}] = \frac{1}{(1-z)^{\Delta_\phi}(1-\bar{z})^{\Delta_\phi}} + \frac{1}{(1+z)^{\Delta_\phi}(1+\bar{z})^{\Delta_\phi}}$$

* And one from the anomalous dimension of ϕ^2

$$\text{Disc}\,[\phi^2] = -\gamma_{\phi^2}\frac{i\pi z(\bar{z} - z\bar{z})}{z-1}\Theta(\bar{z}-1)$$

O(N) model in ε-expansion

With both methods we get:

Operator $\mathcal{O}$	$a_{\mathcal{O}}$ KMS DR	DR	LIF
$\mathbb{1}$	1	0	0
$[\phi\phi]_{0,J\geq 2}$	$-2\zeta'(2+J)-\frac{1}{J}a^{(0)}_{\phi^2}\gamma_{\phi^2}\zeta(J)$	$-\frac{1}{J}a^{(0)}_{\phi^2}\gamma_{\phi^2}$	$-\frac{1}{J}a^{(0)}_{\phi^2}\gamma_{\phi^2}$
$[\phi\phi]_{1,J}$	$\frac{1}{J+2}a^{(0)}_{\phi^2}\gamma_{\phi^2}\zeta(2+J)$	$\frac{1}{J+2}a^{(0)}_{\phi^2}\gamma_{\phi^2}$	$\frac{1}{J+2}a^{(0)}_{\phi^2}\gamma_{\phi^2}$

✱ A new tower of operators $\phi\,\square\,\partial^J\phi \propto \epsilon\phi\partial^J\phi^3$.

✱ We check our results against **Feynman diagram** computation.

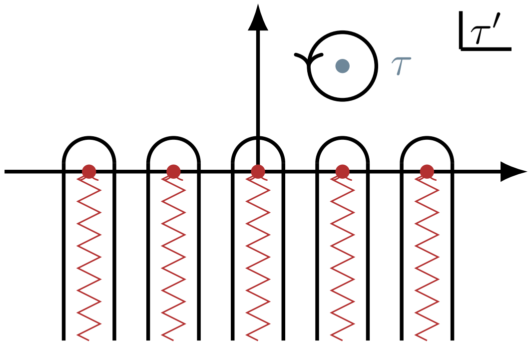
More results with DR

* 2D theories
* 3D O(N) large N

} Infinite number of contributions to discontinuity

* Relations with momentum space. [Andrea Manenti 2019]

Complex τ plane DR



$$g(\tau) = \sum_{m=-\infty}^{\infty} \frac{1}{2\pi i} \int_{-i\infty}^{0} d\tau' \frac{\text{Disc}\, g(\tau')}{\tau' - \tau + \frac{m}{\beta}} + g_{arcs}$$

constant

Computing discontinuity **block by block in OPE** $\text{Disc}\, g(\tau) = \sum_{\Delta} a_\Delta \,\text{Disc}\, \tau^{\Delta - 2\Delta_\phi}$

$$g(\tau) = \sum_{\Delta} \frac{a_\Delta}{\beta^\Delta} \left(\zeta_H \left(2\Delta_\phi - \Delta, \frac{\tau}{\beta} \right) + \zeta_H \left(2\Delta_\phi - \Delta, 1 - \frac{\tau}{\beta} \right) \right) + \kappa$$

expansion in GFF blocks

$$\zeta_H(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}$$

Obeys all bootstrap requirements (OPE, KMS, analyticity & Regge).

Arcs (constant κ - iff $g(\tau)$ bounded) that can be fixed imposing clustering.

Improving Tauberian

Expanding $$g(\tau) = \sum_{\Delta} a_\Delta \underbrace{\Big(\zeta_H(2\Delta_\phi - \Delta, \tau) + \zeta_H(2\Delta_\phi - \Delta, 1 - \tau)\Big)}_{g_{GFF}(\tau)} + \kappa$$

We can exactly compute corrections to Tauberian:

Laplace transform:

$$a_\Delta \overset{\Delta \gg 1}{\sim} 2\frac{\Delta^{2\Delta_\phi - 1}}{\Gamma(2\Delta_\phi)}\left(1 + \frac{\Delta_\phi(1-2\Delta_\phi)}{\Delta} + \frac{(\Delta_\phi - 1)\Delta_\phi(2\Delta_\phi - 1)(6\Delta_\phi - 1)}{6\Delta^2} + \dots\right.$$

Identity

$$+ \frac{a_{\Delta_1}\Gamma(2\Delta_\phi)}{\Gamma(2\Delta_\phi - \Delta_1)}\frac{1}{\Delta^{\Delta_1}}\left(1 + \frac{(\Delta_1 - 2\Delta_\phi + 1)(\Delta_1 + 2\Delta_\phi)}{2\Delta} + \dots\right) + \dots$$

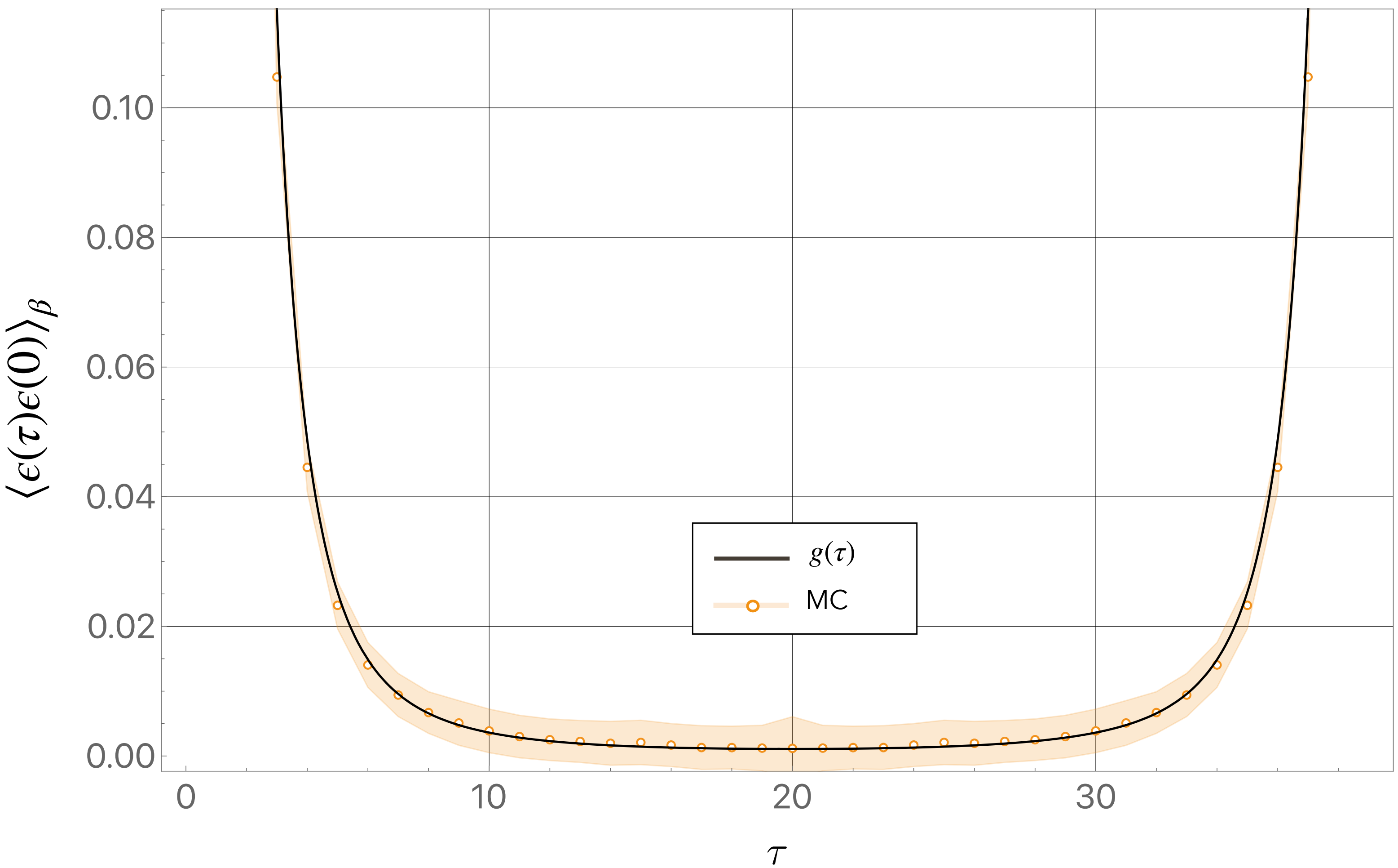
first light operator

$$\left. + \frac{a_{\Delta_2}\Gamma(2\Delta_\phi)}{\Gamma(2\Delta_\phi - \Delta_2)}\frac{1}{\Delta^{\Delta_2}}\left(1 + \frac{(\Delta_2 - 2\Delta_\phi + 1)(\Delta_2 + 2\Delta_\phi)}{2\Delta} + \dots\right) + \dots\right)$$

second light operator

This allows to significantly improve our numerical method!

New numerical results



3d Ising

Holographic CFTs

[2510.20894 Barrat, Bozkurt, Marchetto, Miscioscia, EP]

Holographic CFTs

- CFT on $S^1_\beta \times \mathbb{R}^{d-1}$: Black Brane in AdS

- CFT on $S^1_\beta \times S^{d-1}_R$: Black Hole in AdS

- Einstein gravity coupled to a scalar of mass

$$m^2 = \Delta_\phi(\Delta_\phi - d)$$

Holographic CFT

- The spectrum appearing in the OPE:
  - Identity operator: $\Delta = 0\,, J = 0$
  - Double trace: $\Delta_{n,J} = 2\Delta_\phi + 2n + J$
  - Multi-Stress-tensors: $\Delta_n = dn\,,\ J_n = 0,2,\ldots, dn/2$

- The OPE coefficients:

$$a_{T^{\mu\nu}} = \frac{\pi^4 \Delta_\phi}{120} \qquad a_{T^{\mu\rho}T_\rho^\nu} = \frac{\pi^8 \Delta_\phi (7\Delta_\phi^3 - 23\Delta_\phi^2 + 22\Delta_\phi + 12)}{201600(\Delta_\phi - 3)(\Delta_\phi - 2)}$$

$$a_{T^{\mu\rho}T^{\nu\sigma}} = \frac{\pi^8 \Delta_\phi (7\Delta_\phi^2 + 6\Delta_\phi + 4)}{201600(\Delta_\phi - 2)} \qquad a_{T^{\mu\nu}T_{\mu\nu}} = \frac{\pi^8 \Delta_\phi (7\Delta_\phi^4 - 45\Delta_\phi^3 + 100\Delta_\phi^2 - 80\Delta_\phi + 48)}{201600\,(\Delta_\phi - 4)(\Delta_\phi - 3)(\Delta_\phi - 2)}$$

[Fitzpatrick, Huang 2019]

Bootstrap & Holography

* Naively the discontinuity allows only:

* Identity operator: $\Delta = 0$

* Multi-tensors with: $\Delta_n \leq 2\Delta_\phi$

$$\Delta_\phi = 1 \;:\; g(\tau) = \frac{\pi^2}{\beta^2} \csc^2\left(\frac{\pi\tau}{\beta}\right)$$

$$\Delta_\phi = 2 \;:\; g(\tau) = \frac{\pi^4}{3\,\beta^4} \csc^4\left(\frac{\pi\tau}{\beta}\right)\left[\cos\left(\frac{2\pi\tau}{\beta}\right) + 2\right]$$

$$\Delta_\phi = 3 \;:\; g(\tau) = \frac{\pi^6}{60\,\beta^6} \csc^6\left(\frac{\pi\tau}{\beta}\right)\left[\cos\left(\frac{4\pi\tau}{\beta}\right) + 26\cos\left(\frac{2\pi\tau}{\beta}\right) + 33\right] + a_T \frac{\pi^2}{\beta^6} \csc^2\left(\frac{\pi\tau}{\beta}\right)$$

$$\Delta_\phi = 4 \;:\; g(\tau) = \frac{\pi^8}{2520\,\beta^8} \csc^8\left(\frac{\pi\tau}{\beta}\right)\left[\cos\left(\frac{6\pi\tau}{\beta}\right) + 120\cos\left(\frac{4\pi\tau}{\beta}\right) + 1191\cos\left(\frac{2\pi\tau}{\beta}\right) + 1208\right] + a_T \frac{\pi^4}{3\,\beta^8} \csc^4\left(\frac{\pi\tau}{\beta}\right)\left[\cos\left(\frac{2\pi\tau}{\beta}\right) + 2\right]$$

$$\Delta_\phi = 5 \;:\; g(\tau) = \frac{\pi^{10}}{181440\,\beta^{10}} \csc^{10}\left(\frac{\pi\tau}{\beta}\right)\left[\cos\left(\frac{8\pi\tau}{\beta}\right) + 502\cos\left(\frac{6\pi\tau}{\beta}\right) + 14608\cos\left(\frac{4\pi\tau}{\beta}\right) + 88234\cos\left(\frac{2\pi\tau}{\beta}\right) + 78095\right] + a_T \frac{\pi^6}{60\,\beta^{10}} \csc^6\left(\frac{\pi\tau}{\beta}\right)\left[\cos\left(\frac{4\pi\tau}{\beta}\right) + 26\cos\left(\frac{2\pi\tau}{\beta}\right) + 33\right] + a_{[T^2]} \frac{\pi^2}{\beta^{10}} \csc^2\left(\frac{\pi\tau}{\beta}\right)$$

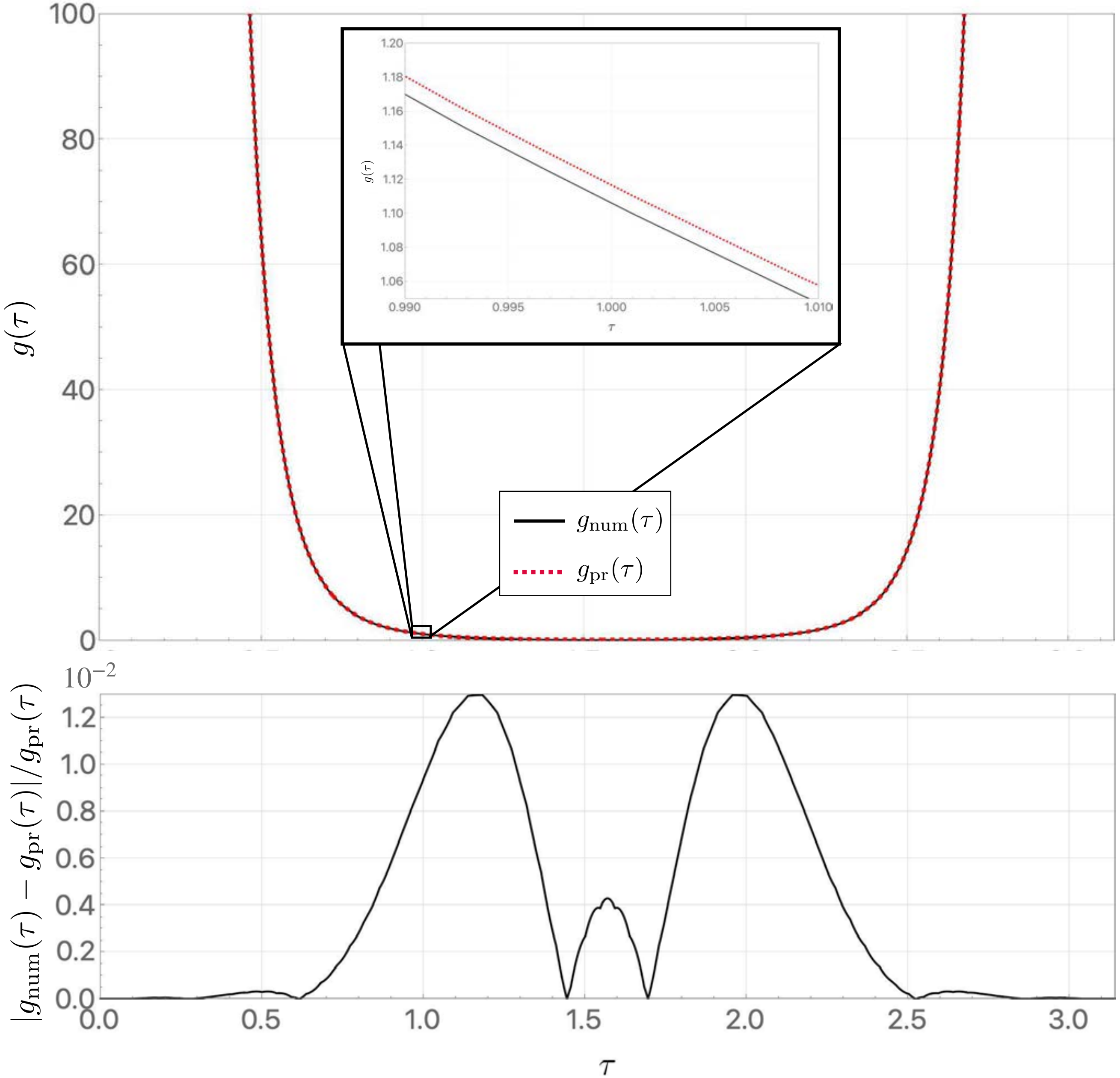
Compare with OPE coefficients of [Fitzpatrick, Huang 2019]

$$a_T = \frac{\pi^4 \Delta_\phi}{40}, \qquad a_{[T^2]} = {a_T}^2 \frac{63\Delta_\phi^4 - 413\Delta_\phi^3 + 672\Delta_\phi^2 - 88\Delta_\phi + 144}{126\Delta_\phi(\Delta_\phi - 4)(\Delta_\phi - 3)(\Delta_\phi - 2)}$$

Compare with numerics

Solving the wave equation in the bulk for $\Delta_\phi = 3$

The discrepancy is always below 1.3%

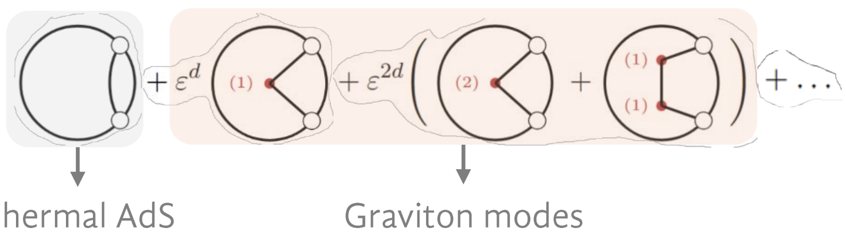


Witten diagrams in the Bulk

A gravitational interpretation of this result:

Expanding the black brane metric around thermal AdS, we can compute the thermal correlator perturbatively in $\varepsilon = L_{AdS}/\beta$

$$\langle \phi(\tau)\phi(0)\rangle_\beta =$$



$$= \sum_n \frac{a_{[T^n]}}{\beta^{dn}} g_{GFF}\left(\Delta_\phi - \frac{dn}{2}, \tau\right)$$

The Witten diagrams reproduce the GFF decomposition.

But there is more…

- The 3pt coefficients have poles: $a_\Delta \sim \frac{1}{\Delta_\phi - n}\,, \quad n \in \mathbb{N}$

- We need to regularise:

$$\frac{1}{\Delta_\phi - n} g_{GFF}\left(\Delta_\phi - \frac{\Delta}{2}, \tau\right) \to \frac{1}{\Delta_\phi + \varepsilon - n} g_{GFF}\left(\Delta_\phi + \varepsilon - \frac{\Delta}{2}, \tau\right)$$

- OPE: Δ degeneracy between some d.t. and m.s. & as $\Delta_\phi \to n$

$$\frac{A}{\Delta_\phi - n}\left(\frac{\tau}{\beta}\right)^{\Delta_{d.t.} - 2\Delta_\phi} - \frac{A}{\Delta_\phi - n}\left(\frac{\tau}{\beta}\right)^{\Delta_{m.s.} - 2\Delta_\phi} = 2A \log\left(\frac{\tau}{\beta}\right)\left(\frac{\tau}{\beta}\right)^{\Delta_{m.s.} - 2\Delta_\phi}$$

- These logs can be seen also from the bulk [Li, Mai, Lü 2019]

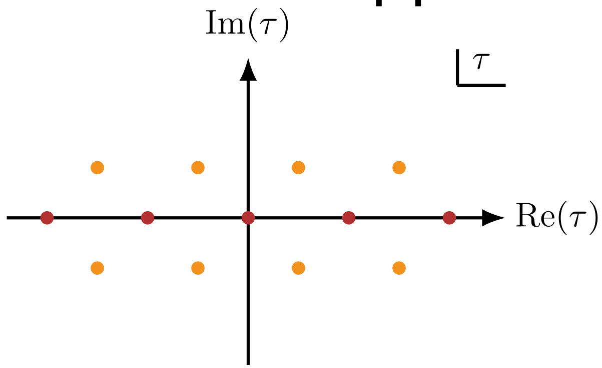
The complete answer

$$g(\tau) = \sum_{\Delta<2\Delta_\phi} \frac{a_\Delta}{\beta^\Delta} g_{GFF}\left(\Delta_\phi - \frac{\Delta}{2}, \tau\right) - 2 \sum_{\Delta\geq 2\Delta_\phi} \frac{\mathrm{Res}_{\Delta_\phi} a_\Delta}{\beta^\Delta} g^{(1,0)}_{GFF}\left(\Delta_\phi - \frac{\Delta}{2}, \tau\right) + g_{arcs}(\tau)$$

* The first part dominates for real τ and we can compute it exactly.

* The regularised part is an infinite sum and we can approximate it.

* It creates new poles on the τ plane.



* They are known to be the bouncing singularities and we remove them using the arcs.

[Čeplak,Liu,Parnachev,Valach 2024]

Conclusions

Where we currently are

✱ **Numerical** approach to the thermal bootstrap.

✱ **Analytical** approach (thermal Dispersion Relation).

} In tandem

✱ Application to Holographic CFTs.

✱ Initiated the $S^1_\beta \times S^{d-1}_R$ geometry for Holographic CFTs.

[2510.20894 Barrat, Bozkurt, Marchetto, Miscioscia, EP]

✱ Temporal line defects (Polyakov loops).

[2407.14600 Barrat, Fiol, Marchetto, Miscioscia, EP]

Where are we going?

- $\mathcal{N}$=4 SYM & ABJM using the spectrum from integrability.
- Apply both approaches to Polyakov loops.
- Study the $S^1 \times S^{d-1}$ geometry.
- Black holes, hydro and CFT data.
- Going away from criticality.

Thank you!