# Super Macdonald polynomials and instanton counting on the blow-up

Based on a collaboration with R.Ohkawa and J.Shiraishi : arXiv [2506.01415]

Hiroaki Kanno

Graduate School of Mathematics & Kobayashi-Maskawa Institute, Nagoya University

International Workshop on QFT and Beyond, 27 November, 2025
Shing-Tung Yau Center of Southeast University (Nanjing)

# Contents

# Macdonald polynomials

$\Lambda_n := \mathbb{Z}[x_1, \ldots, x_n]^{S_n}$ : Ring of the symmetric polynomials in $x_1, \ldots, x_n$.

$$\Lambda_n \ni f(x_1, x_2, \ldots, x_n), \qquad f(x_1, x_2, \ldots, x_n) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}), \quad \sigma \in S_n.$$

We may extend the coefficients to any field $\mathbb{F}$ to define $\Lambda_{\mathbb{F}} := \Lambda_n \otimes_{\mathbb{Z}} \mathbb{F}$.

The Macdonald polynomials $P_\lambda(x; q, t)$ indexed by the partition $\lambda$ give a distinguished basis of $\Lambda_{\mathbb{F}}$, where $\mathbb{F} = \mathbb{Q}(q, t)$; the field of rational functions in parameters $(q, t)$.

1. When $q = t$, $P_\lambda(x; q, t)$ reduce to the Schur polynomials $s_\lambda(x)$.
2. The Macdonald polynomials $P_\lambda(x; q, t)$ are invariant under $(q, t) \to (q^{-1}, t^{-1})$.

# Pieri formula for the Macdonald polynomials

The Pieri formula for the multiplication of $P_{\square} = p_1 = x_1 + x_2 + \cdots$;

$$p_1 \cdot P_\lambda(x;q,t) = \sum_\mu \psi'_{\mu/\lambda}(q,t) P_\mu(x;q,t),$$

where the sum is over the partitions $\mu$ such that $\mu_k = \lambda_k + \delta_{kj}$ for some $j$ and the expansion coefficients are

$$\psi'_{\mu/\lambda}(q,t) = \prod_{i=1}^{j-1} \frac{(1-q^{\lambda_i-\lambda_j}t^{j-i-1})(1-q^{\lambda_i-\lambda_j-1}t^{j-i+1})}{(1-q^{\lambda_i-\lambda_j}t^{j-i})(1-q^{\lambda_i-\lambda_j-1}t^{j-i})}, \qquad \mu = \lambda + \mathbf{1}_j.$$

The simplest example is

$$P_{\square}(x;q,t) \cdot P_{\square}(x;q,t) = p_1^2 = P_{\square\square}(x;q,t) + \frac{(1-q)(1-t^2)}{(1-qt)(1-t)} P_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}(x;q,t).$$

When $q = t$, namely for Schur polynomials $s_\lambda(x)$, $\psi'_{\mu/\lambda}(q,q) = 1$.

# Relation to the $\mathfrak{gl}_1$ quantum toroidal algebra

Macdonald polynomials $P_\lambda(x;q,t)$ provide a basis of a level zero representation of the $\mathfrak{gl}_1$ quantum toroidal algebra (DIM algebra) with the generating currents;

$$E(z)=\sum_{k\in\mathbb{Z}} E_k z^{-k}, \quad F(z)=\sum_{k\in\mathbb{Z}} F_k z^{-k}, \qquad K^\pm(z)=\sum_{n=0}^{\infty} K^\pm_{\pm n} z^{\mp n}.$$

Level zero representation $\Longrightarrow$ the Cartan modes $K^\pm_{\pm n}$ are mutually commuting.

$|\lambda) := P_\lambda(x;q,t)$: simultaneous eigenstates of $K^\pm_{\pm n}$; $\quad \ell(\lambda)$: the length of the partition $\lambda$,

$$K^-(z)|\lambda) = \prod_{s=1}^{\ell(\lambda)} \psi(q_3 x_s^{-1} z/v)^{-1} \prod_{s=1}^{\ell(\lambda)+1} \psi(q_1^{-1} x_s^{-1} z/v)|\lambda), \qquad \psi(z) := \frac{q_3^{\frac{1}{2}} - q_3^{-\frac{1}{2}} z}{1-z},$$

where $(q,t)=(q_1,q_2^{-1})$ with $q_1q_2q_3=1$ and $x_s := q_1^{\lambda_s-1} q_2^{s-1}$.

Actions of the raising and lowering operators;

$$E(z)|\lambda) = \frac{1}{1-q_1}\sum_{j=1}^{\ell(\lambda)+1}\prod_{s=1}^{j-1}\psi(q_1^{-1}x_s/x_j)\psi(q_3x_s/x_j)^{-1}\delta(q_1x_jv/z)|\lambda+\mathbf{1}_j),$$

$$F(z)|\lambda) = \frac{1}{1-q_1^{-1}}\sum_{j=1}^{\ell(\lambda)}\prod_{s=j+1}^{\ell(\lambda)}\psi(x_s/x_j)\prod_{s=j+1}^{\ell(\lambda)+1}\psi(q_2^{-1}x_s/x_j)^{-1}\delta(x_jv/z)|\lambda-\mathbf{1}_j).$$

Up to the normalization factor $\frac{1}{1-q_1}$, the action of the zero mode $E_0$ gives the Pieri rule;

$$(1-q_1)E_0|\lambda) = \sum_{j=1}^{\ell(\lambda)+1}\prod_{s=1}^{j-1}\frac{(1-q_2(x_s/x_j))(1-q_3(x_s/x_j))}{(1-q_1^{-1}(x_s/x_j))(1-(x_s/x_j))}|\lambda+\mathbf{1}_j).$$

$$x_s/x_j = q_1^{\lambda_s-\lambda_j}q_2^{s-j} = q^{\lambda_s-\lambda_j}t^{j-s}.$$

# Relation to the instanton counting

The fixed points of the torus action on the moduli space $\mathrm{Hilb}^{[*]}(\mathbb{C}^2)$ of $U(1)$ instantons are labeled by the partitions. From the equivariant character for a pair of fixed points $(\lambda, \mu)$;

$$\chi_{(\lambda,\mu)}(u|t_1,t_2) = u \cdot \left[ \sum_{s\in\lambda} t_1^{-\ell_\mu(s)} t_2^{a_\lambda(s)+1} + \sum_{t\in\mu} t_1^{\ell_\lambda(t)+1} t_2^{-a_\mu(t)} \right], \qquad (t_1,t_2) = (t, q^{-1}),$$

Nakajima-Yoshioka (2003)

we obtain the Nekrasov factor

$$\mathsf{N}_{\lambda\mu}(u|q,t) = \prod_{(i,j)\in\lambda} (1 - uq^{-\lambda_i+j-1} t^{-\mu_j^\vee+i}) \prod_{(i,j)\in\mu} (1 - uq^{\mu_i-j} t^{\lambda_j^\vee-i+1})$$

$$= \prod_{i,j=1}^{\infty} \frac{(uq^{\mu_i-\lambda_j} t^{j-i+1}; q)_\infty (ut^{j-i}; q)_\infty}{(uq^{\mu_i-\lambda_j} t^{j-i}; q)_\infty (ut^{j-i+1}; q)_\infty}.$$

which is a basic building block of the instanton (Nekrasov) partition function.

We define the integral form of the Macdonald polynomial by $J_\lambda = c_\lambda P_\lambda$, where $c_\lambda$ appears in the norm of the Macdonald polynomial $\langle P_\lambda | P_\lambda \rangle = \frac{c'_\lambda}{c_\lambda}$, with

$$c_\lambda := \prod_{s \in \lambda} (1 - q^{a(s)} t^{\ell(s)+1}), \qquad c'_\lambda := \prod_{s \in \lambda} (1 - q^{a(s)+1} t^{\ell(s)}).$$

The Pieri coefficients for the integral form are

$$\psi'_{(\lambda + \mathbf{1}_k)/\lambda} \cdot \frac{c_\lambda}{c_{\lambda + \mathbf{1}_k}} = t^{1-k} \prod_{\substack{j=1 \\ j \neq k}}^{\infty} \frac{1 - q^{\lambda_k - \lambda_j} t^{j-k+1}}{1 - q^{\lambda_k - \lambda_j} t^{j-k}}.$$

On the other hand, from the infinite product form of the Nekrasov factor, we compute

$$\frac{1}{1-t} \mathrm{Res}_{u=0} \left( \frac{\mathsf{N}_{\lambda,\lambda}(u|q,t)}{\mathsf{N}_{\lambda,\lambda+\mathbf{1}_k}(u|q,t)} \right) = \prod_{\substack{j=1 \\ j \neq k}}^{\infty} \frac{1 - q^{\lambda_k - \lambda_j} t^{j-k+1}}{1 - q^{\lambda_k - \lambda_j} t^{j-k}}.$$

The discrepancy of a monomial factor is adjusted by introducing the factor $t^{n(\lambda)}$ with

$$n(\lambda) := \sum_{s \in \lambda} \ell_\lambda(s) = \sum_{(i,j) \in \lambda} (i-1) = \sum_{i=1}^{\infty} (i-1)\lambda_i, \qquad n(\lambda + \mathbf{1}_k) - n(\lambda) = k - 1.$$

In summary we have [Feigin-Tsymbaliuk (2009)]

$$\boxed{\frac{1}{1-t} \mathrm{Res}_{u=0} \left( \frac{\mathsf{N}_{\lambda,\lambda}(u|q,t)}{\mathsf{N}_{\lambda,\lambda+\mathbf{1}_k}(u|q,t)} \right) = \psi'_{(\lambda+\mathbf{1}_k)/\lambda} \cdot \frac{c_\lambda \cdot t^{n(\lambda+\mathbf{1}_k)}}{c_{\lambda+\mathbf{1}_k} \cdot t^{n(\lambda)}}}$$

The left hand side allows a natural interpretation by the action of the DIM algebra on the $K$ theory group of $\mathrm{Hilb}^{[*]}(\mathbb{C}^2)$, which is constructed by the method of “correspondence”.

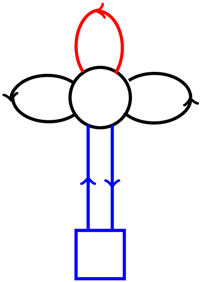
We are going to generalize this story to the case of supersymmetric Macdonald polynomials.

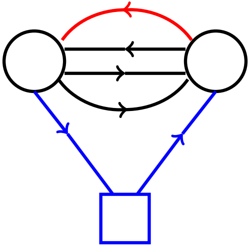
# Main message of the talk

We are going to show the following generalization;

| | |
|---|---|
| Macdonald polynomial $P_\lambda(x;q,t)$ | Super Macdonald polynomial $\mathcal{M}_\Lambda(x,\theta;q,t)$ |
| Partition: Young diagram $\lambda$ | Super partition: super Young diagram $\Lambda$ |
| Fock space of a free boson | Fock space of (a free boson $\otimes$ a free NS fermion) |
| Quantum toroidal algebra of type $\mathfrak{gl}_1$ | Quantum toroidal algebra of type $\mathfrak{gl}_{1\|1}$ |
| Instanton counting on $\mathbb{C}^2$ or $\mathbb{P}^2$ | Instanton counting on $\widehat{\mathbb{C}^2}$ or $\widehat{\mathbb{P}^2}$ (blow-ups) |
| BPS states on $\mathbb{C}^3$ | BPS states on the resolved conifold |

# Description in terms of the quiver with potential





$$W = \mathrm{Tr}\left( B_3[B_1, B_2] + B_3 IJ \right) \qquad W = \mathrm{Tr}\left( A_1 B_1 A_2 B_2 + A_1 B_2 A_2 B_1 + A_1 IJ \right)$$

Delete the framing (the blue part) $\Longrightarrow$ Quiver data for the quantum toroidal algebra

Delete the red edge $\Longrightarrow$ Quiver data for the instanton moduli space

# Contents

# Super Macdonald polynomials

The super Macdonald polynomials $\mathcal{M}_\Lambda(x,\theta;q,t)$ generalize the Macdonald polynomials to the super space with additional Grassmann coordinates $\theta_i$. They are invariant under *simultaneous* permutations of the commuting and the anti-commuting coordinates;

$$\mathcal{M}_\Lambda(x_i,\theta_i;q,t) = \mathcal{M}_\Lambda(x_{\sigma(i)},\theta_{\sigma(i)};q,t), \quad \sigma \in S_n.$$

The super Macdonald polynomials $\mathcal{M}_\Lambda(x,\theta;q,t)$ are indexed by the set of super partitions.
[Blondeau-Fournier, Desrosiers, Lappinte, Mathieu (2011-12)]

A super partition is a non-increasing sequence of non-negative elements in $\mathbb{Z}/2$;

$$\Lambda_1 \geq \Lambda_2 \geq \ldots \geq \Lambda_i \geq \ldots \geq \Lambda_{\ell(\Lambda)} > 0, \qquad \Lambda_i \in (\mathbb{Z}/2)_{\geq 0},$$

where $\ell(\Lambda)$ is the number of non-zero components in $\Lambda$.

$\mathbb{Z}_2$ grading of $\mathbb{Z}/2$ : integral elements are even and non-integral elements are odd.
We require that if $\Lambda_k$ is odd ($\Lambda_k \in \mathbb{Z}+\frac{1}{2}$), the inequality is strict $\Lambda_{k-1} > \Lambda_k > \Lambda_{k+1}$.

# Super Macdonald polynomials up to $|\Lambda| \le 2$

In terms of the bosonic and the fermionic power sum polynomials defined by

$$p_k := \sum_{i=1}^{n} x_i^k, \qquad \pi_k := \sum_{i=1}^{n} \theta_i x_i^{k-1},$$

we have

$$\mathcal{M}_{(\frac{1}{2})} = \pi_1, \qquad \mathcal{M}_{(1)} = p_1, \qquad \mathcal{M}_{(\frac{3}{2})} = \frac{q(1-t)}{1-qt} p_1 \pi_1 + \frac{1-q}{1-qt} \pi_2,$$

$$\mathcal{M}_{(1,\frac{1}{2})} = p_1 \pi_1 - \pi_2, \qquad \mathcal{M}_{(2)} = \frac{1}{2} \left( \frac{(1+q)(1-t)}{1-qt} p_1^2 + \frac{(1+t)(1-q)}{1-qt} p_2 \right),$$

$$\mathcal{M}_{(1,1)} = \frac{1}{2}(p_1^2 - p_2), \qquad \mathcal{M}_{(\frac{3}{2},\frac{1}{2})} = \pi_2 \pi_1.$$

1. Even if we set $q = t$, there remains one parameter in the super "Schur" polynomials.
2. The super Macdonald polynomials $\mathcal{M}_\Lambda(x, \theta; q, t)$ are not invariant under $(q,t) \to (q^{-1}, t^{-1})$.

# Super Young diagram and Fock space

It is convenient to identify a super partition with a super Young diagram, which consists of full boxes and upper half boxes. If a row is even, the end of the row is a full box $\square$ and if it is odd, the end of the row is an upper half box $\boxtimes$ which is counted as $\frac{1}{2}$.

The generating function of the numbers of super partitions is

$$\sum_{\Lambda} x^n y^m = \prod_{k=1}^{\infty} \frac{1+yx^{k-1}}{1-x^k},$$

where $n$ is the number of $\square$ and $m$ is the number of $\boxtimes$. With $|\Lambda| = n + \frac{m}{2}$, we have

$$\sum_{\Lambda} q^{2|\Lambda|} = \prod_{k=1}^{\infty} \frac{1+q^{2k-1}}{1-q^{2k}}.$$

This is nothing but the character of the tensor product of the Fock spaces of a free boson $a_n$ and a free (NS) fermions $\psi_r$.

The bosonic and the fermionic creation operators $a_{-n}$ $(n \in \mathbb{Z}_{>0})$ and $\psi_{-r}$ $(r \in \mathbb{Z}_{\geq 0} + \frac{1}{2})$ correspond to a bosonic row of length $n$ and a fermionic row of length $r$, respectively.

We also employ the following descriptions of super partitions;

1. $\Lambda = (\lambda, \overline{\sigma})$, where $\lambda$ is a partition and $\overline{\sigma} = (\sigma_i),\ \sigma_i \in \{0, 1\}$.

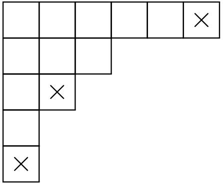
$$\Lambda_i = \lambda_i - \frac{1}{2}\sigma_i.$$

   The $k$-th row is bosonic or fermionic according to $\sigma_k = 0$ or $\sigma_k = 1$.

2. $\Lambda = (Y, S)$ by a Young diagram $Y$ and a *subset* $S$ of removable boxes in $Y$.
   We call elements in $S$ marked boxes, which are identified with $\boxtimes$.

For a pair of two marked boxes $\mathsf{s} \neq \mathsf{s}'$, we associate an irrelevant box in $Y$.
If two end points of the hook of a box in $Y$ are both marked, we call the box irrelevant.

We will include the marked boxes to the set of irrelevant boxes and define the set of relevant boxes as the compliment of the set of irrelevant boxes. The number of the irrelevant boxes is $\frac{1}{2}m(m+1)$.



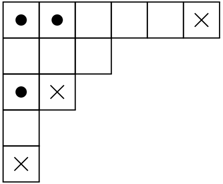


Figure: Left: Super partition $\Lambda = (\frac{11}{2}, 3, \frac{3}{2}, 1, \frac{1}{2})$ with fermion number $m = 3$. The marked boxes are counted as $\frac{1}{2}$. For each row and each column the number of the marked box is at most one.
Right: For each pair of marked boxes we associate a box with $\bullet$. The irrelevant boxes are boxes with $\times$ or $\bullet$. The number of irrelevant boxes is $3 + \frac{3 \cdot 2}{2} = 6$.

We employ the notations $Y^{\circledast} = Y$ and $Y^{*} = Y \setminus S$. We also use the notations $\Lambda^{\circledast}$ and $\Lambda^{*}$. Both $\Lambda^{\circledast}$ and $\Lambda^{*}$ are ordinary Young diagrams such that the skew diagram $\Lambda^{\circledast} \setminus \Lambda^{*} = S$.

# Contents

# Quantum toroidal algebra of type $\mathfrak{gl}_{1|1}$

One of the ways to define the super Macdonald polynomials is to use the representation theory of the quantum toroidal algebra $\mathcal{U}_{(q,t)}(\widehat{\widehat{\mathfrak{gl}}}_{1|1})$, which is a quantum group (we have a coproduct $\Delta$) with generators $E_{i,k}, F_{i,k}, K^{\pm}_{i,\pm r}$ $(i = 1, 2)$ and a central element $C$.
[Galakhov-Morozov-Tselousov (2024)]

Introduce the generating currents;

$$E_i(z) = \sum_{k\in\mathbb{Z}} E_{i,k} z^{-k}, \qquad F_i(z) = \sum_{k\in\mathbb{Z}} F_{i,k} z^{-k}, \qquad K_i^{\pm}(z) = \sum_{r\geq 0} K^{\pm}_{i,\pm r} z^{\mp r}.$$

$E_i(z)$ and $F_i(z)$ are odd currents.

Level zero representations: $C = 1 \Longrightarrow K^{\pm}_{i,\pm r}$ are mutually commuting.

Super Macdonald polynomials are simultaneous eigenstates of $K^{\pm}_{i,\pm r}$ .

Other commutation relations are (the shift parameters $r_i$ appear in general).

$$K_i^{\pm}(z)E_j(w) = \varphi^{j\Rightarrow i}(z,w)E_j(w)K_i^{\pm}(z), \quad K_i^{\pm}(z)F_j(w) = \varphi^{j\Rightarrow i}(z,w)^{-1}F_j(w)K_i^{\pm}(z),$$

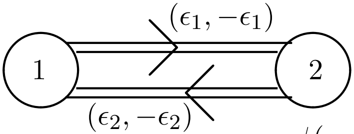
$$[E_i(z), F_j(w)] = \delta_{ij}\delta\left(\frac{w}{z}\right)\left(z^{r_i}K_i^{+}(z) - K_i^{-}(w)\right),$$

$$E_i(z)E_j(w) = (-1)\cdot\varphi^{j\Rightarrow i}(z,w)E_j(w)E_i(z), \quad F_i(z)F_j(w) = (-1)\cdot\varphi^{j\Rightarrow i}(z,w)^{-1}F_j(w)F_i(z).$$

The structure function $\varphi^{i\Rightarrow j}(z,w)$ is determined by the quiver data;

[Galakhov-Li-Yamazaki (2021), Noshita-Watanabe (2021)]

$$\varphi^{i\Rightarrow j}(z,w) = \frac{\prod_{I\in\{j\to i\}}\phi(q_I;z,w)}{\prod_{I\in\{i\to j\}}\phi(q_I^{-1};z,w)}, \qquad \phi(p;z,w) := p^{\frac{1}{2}}z - p^{-\frac{1}{2}}w = p^{\frac{1}{2}}(z-p^{-1}w).$$

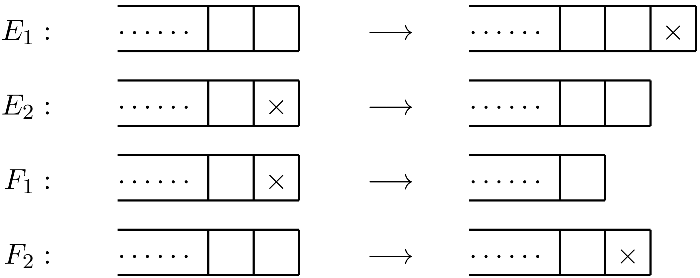



$$\varphi^{1\Rightarrow 1}(z,w) = \varphi^{2\Rightarrow 2}(z,w) = 1.$$

$$\varphi^{1\Rightarrow 2}(z,w) = \varphi^{2\Rightarrow 1}(z,w)^{-1} = \frac{\phi(q_2;z,w)\phi(q_2^{-1};z,w)}{\phi(q_1;z,w)\phi(q_1^{-1};z,w)} = \frac{(z-q_2w)(z-q_2^{-1}w)}{(z-q_1w)(z-q_1^{-1}w)}.$$

## Super Fock representation

We can construct a level zero representation on the vector space spanned by super partitions. The action of odd currents $E_i(z)$ and $F_i(z)$ on the super partitions is described as follows;




$$E_{1,2} : |\Lambda| \longrightarrow |\Lambda| + \tfrac{1}{2} \text{ and } F_{1,2} : |\Lambda| \longrightarrow |\Lambda| - \tfrac{1}{2}.$$

$E_1, F_2$ : boson $\longrightarrow$ fermion, while $E_2, F_1$ : fermion $\longrightarrow$ boson.

## Generating functions of the eigenvalues

In the super Fock representation the Cartan generators $K^{\pm}_{i,\pm r}$ are simultaneously diagonalized on the set of super partitions. The Cartan currents $K^{\pm}_{i}(z)$ are nothing but the generating functions of the eigenvalues.

$$K_1^{\pm}(z) = \left[ \frac{1}{z - t^{-\ell(\Lambda)}} \prod_{i=1}^{\ell(\Lambda)} \frac{z - q^{\lambda_i - \overline{\sigma}_i} t^{-i}}{z - q^{\lambda_i - \overline{\sigma}_i} t^{1-i}} \right]_{\pm},$$

$$K_2^{\pm}(z) = \left[ (z - (t/q)^{\frac{1}{2}} t^{-\ell(\Lambda)}) \prod_{i=1}^{\ell(\Lambda)} \frac{z - q^{\lambda_i - \frac{1}{2}} t^{\frac{3}{2} - i}}{z - q^{\lambda_i - \frac{1}{2}} t^{\frac{1}{2} - i}} \right]_{\pm},$$

where $[f(z)]_{\pm}$ denotes the (Laurent) expansion of $f(z)$ around $z = \infty$ or $z = 0$. Since the numbers of factors in the denominator and the numerator are not balanced, we need the shift parameters $r_1 = -1$ and $r_2 = 1$, which is a result of the appropriate regularization of a semi-infinite tensor product.

# Pieri rule from the action of the zero modes $E_{i,0}$

$$E_{i,0}|\lambda, \overline{\sigma}\rangle = \sum_{k=1}^{\ell(\lambda)+1} \psi_k^{(i)}(q,t) \cdot \overline{\delta}_{i+\overline{\sigma}_k,1} |\lambda + 1_k \cdot \delta_{i,1}, \overline{\sigma} + \overline{1_k}\rangle.$$

The coefficients of the Pieri rule are

$$\psi_k^{(1)}(q,t) = (-1)^{F(k)} \prod_{i=1}^{k-1} \frac{1 - q^{\lambda_i - \lambda_k - \overline{\sigma}_i} t^{k-i-1}}{1 - q^{\lambda_i - \lambda_k - \overline{\sigma}_i} t^{k-i}},$$

$$\psi_k^{(2)}(q,t) = (1-t)(-1)^{F(k)} \prod_{i=1}^{k-1} \frac{1 - q^{\lambda_i - \lambda_k} t^{1+k-i}}{1 - q^{\lambda_i - \lambda_k} t^{k-i}},$$

where $F(k) := \sum_{i=1}^{k-1} \overline{\sigma}_i$ is the number of fermionic rows above the $k$-th row.

The sign factor $(-1)^{F(k)}$ comes from the fermionic nature of $E_i(z)$.

# Contents

# Quiver description of the instanton moduli space on the blow up

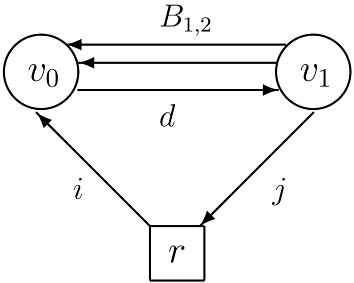



Figure: ADHM-like quiver for the framed sheaves $E$ on the blow-up with the exceptional curve $C$.

$$rk(E) := r = \dim W, \qquad v_0 := \dim V_0 \text{ and } v_1 := \dim V_1.$$

$$k := -(c_1(E), C) = v_1 - v_0, \qquad n := (c_2(E) - (r-1)c_1(E)^2/(2r), \widehat{\mathbb{P}}^2), \qquad n + \frac{k^2}{2r} = \frac{1}{2}(v_0 + v_1).$$

$$\mu(B_1, B_2, d, i, j) := B_1 d B_2 - B_2 d B_1 + ij = 0, \qquad \text{F-term condition}$$

# Torus fixed points and Nekrasov factor

The torus fixed points in the moduli space are in bijection to $r$-tuples of pairs $(Y_\alpha, S_\alpha)$ of a Young diagram $Y_\alpha$ and a subset $S_\alpha$ of removable boxes such that

$$\sum_\alpha |S_\alpha| = (c_1, [C]), \qquad \sum_\alpha |Y_\alpha| = -\int_{\widehat{\mathbb{P}^2}} \mathrm{ch}_2 + \frac{1}{2}(c_1, [C]).$$

The equivariant character for a pair of super partitions $\Lambda_\alpha = (Y_\alpha, S_\alpha)$ and $\Lambda_\beta = (Y_\beta, S_\beta)$ is computed as follows;

$$\chi_{(\Lambda_\alpha,\Lambda_\beta)}(q,t) = \sum_{s\in Y_\alpha^*} t^{-\ell_{Y_\beta^{\circledast}}(s)} q^{-a_{Y_\alpha^*}(s)-1} + \sum_{t\in Y_\beta^{\circledast}} t^{\ell_{Y_\alpha^*}(t)+1} q^{a_{Y_\beta^{\circledast}}(t)} - \sum_{s\in S_\alpha, t\in S_\beta} t^{\ell'(s)-\ell'(t)} q^{a'(t)-a'(s)},$$

Nakajima-Yoshioka (2008)

where $\ell'(i,j) = i-1,\ a'(i,j) = j-1$ are co-leg length and co-arm length, respectively.

From the formula of $\chi_{(\Lambda_\alpha,\Lambda_\beta)}(q,t)$, we obtain the Nekrasov factor for the instanton counting on the blow-up;

$$
\begin{aligned}
&\mathsf{N}_{\Lambda_\alpha,\Lambda_\beta}(q,t)\\
&=\prod_{i,j=1}^{\infty}\frac{(q^{(Y_\beta^{\circledast})_j-(Y_\alpha^{*})_i}t^{i-j+1};q)_\infty(t^{i-j};q)_\infty}{(q^{(Y_\beta^{\circledast})_j-(Y_\alpha^{*})_i}t^{i-j};q)_\infty(t^{i-j+1};q)_\infty}\cdot\prod_{s\in S_\alpha,t\in S_\beta}\frac{1}{1-t^{\ell'(s)-\ell'(t)}q^{a'(t)-a'(s)}}.
\end{aligned}
$$

Assume that the $k$-th low of $\Lambda$ is bosonic. Let us add a marked box to this low, which makes the row fermionic. The change of the Nekrasov factor under this operation is

$$
\frac{\mathsf{N}_{\Lambda,\Lambda}(u|q,t)}{\mathsf{N}_{\Lambda,\Lambda+\boxtimes_k}(u|q,t)}=\frac{1-ut}{1-u}\prod_{\substack{i=1\\i\neq k}}^{\infty}\frac{1-uq^{\lambda_k-\lambda_i+\overline{\sigma}_i}t^{i-k+1}}{(1-uq^{\lambda_k-\lambda_i}t^{i-k})^{\delta_{\overline{\sigma}_i,0}}}.
$$

Note that the denominator becomes trivial for fermionic rows of $\Lambda$.

Similarly we assume that the $k$-th row of $\Lambda$ is fermionic. Namely the last box of the row is marked. Let us remove the marking, which makes the row bosonic.
The change of the Nekrasov factor under this operation is

$$\frac{\mathsf{N}_{\Lambda,\Lambda}(u|q,t)}{\mathsf{N}_{\Lambda,\Lambda(\boxtimes_k\to\square_k)}(u|q,t)} = \frac{1}{1-u}\prod_{\substack{i\in F\\ i\neq k}} \frac{1}{1-uq^{\lambda_k-\lambda_i}t^{i-k}}.$$

In this case, only the fermionic rows contribute to the matrix element.

The factor

$$\widetilde{c}_\Lambda(q,t) := \prod_{s\in\mathcal{B}(\Lambda)} (1-t^{\ell_{\Lambda^*}(s)+1}q^{a_{\Lambda^{\circledast}}(s)}),$$

where $\mathcal{B}(\Lambda)$ denotes the set of the relevant boxes, appears in the conjecture for the norm of super Macdonald polynomials. It is natural to define the integral form of the super Macdonald polynomial by

$$\mathcal{J}_\Lambda := \widetilde{c}_\Lambda(q,t)\mathcal{M}_\Lambda.$$

# Base change to the super Macdonald basis

By identifying the fixed point bases of the $K$-theory group of the instanton moduli space with the integral form. we have

$$\mathrm{Res}_{u=1}\left(\frac{\mathsf{N}_{\Lambda,\Lambda}(u|q,t)}{\mathsf{N}_{\Lambda,\Lambda+\boxtimes_k}(u|q,t)}\right)\frac{\widetilde{c}_{\Lambda+\boxtimes_k}(q,t)}{\widetilde{c}_{\Lambda}(q,t)}\frac{t^{n(\Lambda^{\circledast})}}{t^{n(\Lambda^{\circledast}+1_k)}}=(-1)^{F(k)}(1-t)\prod_{i=1}^{k-1}\frac{1-t^{k-i-1}q^{\lambda_i-\lambda_k-\overline{\sigma}_i}}{1-t^{k-i}q^{\lambda_i-\lambda_k-\overline{\sigma}_i}},$$

$$\mathrm{Res}_{u=1}\left(\frac{\mathsf{N}_{\Lambda,\Lambda}(u|q,t)}{\mathsf{N}_{\Lambda,\Lambda(\boxtimes_k\to\square_k)}(u|q,t)}\right)\frac{\widetilde{c}_{\Lambda(\boxtimes_k\to\square_k)}(q,t)}{\widetilde{c}_{\Lambda}(q,t)}=(-1)^{F(k)}(1-t)\prod_{i=1}^{k-1}\frac{1-t^{k-i+1}q^{\lambda_i-\lambda_k}}{1-t^{k-i}q^{\lambda_i-\lambda_k}}.$$

Up to the factor $1-t$ the right hand sides are the Pieri coefficients for the super Macdonald polynomials.