Double Affine Hecke Algebra & Skein Algebra

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Outline

Quantum Invariant: volume conjecture, QMF, DAHA

Quantum invariants (Jones polynomial, HOMFLY polynomial, WRT invariant, …)

- Geometry: Volume conjecture for Kashaev invariant $\langle K \rangle_N = J_N(K; \zeta_N)$ (the equality with the colored Jones poly is due to Murakami-Murakami)

$$\lim_{N\to\infty} \frac{2\pi}{N} \log \left| \langle K \rangle_N \right| = \mathrm{Vol}(S^3 \setminus K) \qquad \zeta_N = e^{2\pi \mathrm{i}/N}$$

- Number theory: Quantum modular form
  - Kontsevich–Zagier series (Kashaev invariant for $T_{3,2}$)
  - Lawrence–Zagier (WRT invariant for Poincaré homology sphere $\Sigma_{2,3,5}$)

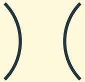
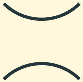
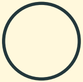
Motivation: Study geometric/number theoretic aspects of refined quantum invariants such as superpolynomial $P_R(K; a, q, t)$ proposed by Dunfield–Gukov–Rasmussen.
Cherednik constructed $P_R(T_{s,t}; a, q, t)$ using DAHA (double affine Hecke algebra)

Problem: 🤔 Construct generalized DAHA for hyperbolic knots

Quantum invariants: volume conjecture & QMF

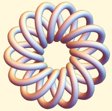
Skein algebra & Jones polynomial

The Jones polynomial $J_2(K;q)$ for knot K are given from the skein algebra

 $= A$  $+ A^{-1}$  ,  $= -A^2 - A^{-2}$

The colored Jones polynomial $J_N(K;q)$ is defined from
N-dim rep of $\mathcal{U}_q(sl_2)$. For $K = T_{s,t}$ Rosso–Jones obtained

$T_{3,14}$



$$J_N(T_{s,t};q) = \frac{q^{\frac{1}{4}st(1-N^2)}}{q^{N/2} - q^{-N/2}} \sum_{r=-(N-1)/2}^{(N-1)/2} \left(q^{str^2-(s+t)r+\frac{1}{2}} - q^{str^2-(s-t)r-\frac{1}{2}} \right)$$

It is a finite sum, and similar to the character of log-VOA.
For some cases, we have q-hypergeometric series expression e.g. $q = e^{2\pi i\tau}$

$$J_N(T_{2,3};q) = q^{1-N} \sum_{n=0}^{\infty} q^{-nN} (q^{1-N})_n, \qquad (x)_n = \prod_{i=1}^{N} (1 - xq^{i-1})$$

which shows $\langle T_{2,3} \rangle_N = \zeta_N F\left({}^1\!/_N\right)$ where the Kontsevich–Zagier series $F(\tau) = \sum_{n=0}^{\infty} (q)_n$.

Zagier (2001) showed that $F(\tau)$ is a typical example of quantum modular forms, i.e., $f : \mathbb{Q} \to \mathbb{C}$ s.t. the function

$$h_\gamma(x) := f(x) - \chi(\gamma)(cx+d)^{-k} f\left(\frac{ax+b}{cx+d}\right), \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \subset SL(2;\mathbb{Z})$$

has "some properties" of continuity or analyticity.

Quantum modular form: number theory from quantum invariants

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In fact, using the "strange identity" $F(\tau) = \sum_{n=0}^{\infty} (q)_n \doteq \sum_{n=0}^{\infty} n\chi_{12}(n) q^{\frac{n^2-1}{24}}$ (Eichler integral of the Dedekind η-function), S-transformation is given as

$$\zeta_{24/N} F(1/N) \simeq -\left(\frac{N}{\mathrm{i}}\right)^{3/2} \mathrm{e}^{-\frac{\pi \mathrm{i}}{12} N} F(-N) + \sum_{k=0}^{\infty} \frac{T(k)}{k!} \left(\frac{\pi}{12 \mathrm{i} N}\right)^k, \qquad \frac{\sin(2x)\sin(3x)}{\sin(6x)} = \sum_{n=0}^{\infty} \frac{T(n)}{(2n+1)!} x^{2n+1}$$

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The volume conjecture suggests that $J_N(T_{s,t};q)$ is also QMF. Indeed we have $\langle T_{s,t}\rangle_N = e^{\frac{s^2t^2-s^2-t^2}{2stN}\pi \mathrm{i}} \widetilde{\Phi}^{(s-1,1)}_{s,t}(1/N)$ which fulfills

$$\widetilde{\Phi}^{(n,m)}_{s,t}(z) + (\mathrm{i}z)^{-3/2} \sum_{n',m'} S^{n',m'}_{n,m} \widetilde{\Phi}^{(n',m')}_{s,t}\left(-\frac{1}{z}\right) = \sqrt{\frac{st\mathrm{i}}{8\pi^2}} \int_0^{\mathrm{i}\infty} \frac{\Phi^{(n,m)}_{s,t}(\tau)}{(\tau - z)^{3/2}} d\tau$$

where $\frac{\Phi^{(n,m)}_{s,t}(\tau)}{\eta(\tau)}$ is the character of Virasoro minimal model, and $S^{n',m'}_{n,m}$ is the S-matrix thereof.

The S-transformation for $\langle T_{s,t} \rangle_N$ is analogous to a weight $3/2$ mock modular forms. The Ramanujan mock theta functions are weight $1/2$ MMF. In quantum topology, they are related to the WRT invariant $\tau_N(M)$ via Habiro's unified WRT invariant $I_q(M)$

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For examples, the WRT for the Brieskorn homology sphere $M = \Sigma_{p_1,p_2,p_3}$ are

$$1 + q(1-q)\, I_q(\Sigma_{2,3,5}) = \sum_{n=0}^{\infty} q^n (q^n)_n = 1 + q + q^3 + q^7 - q^8 - q^{14} - q^{20} - q^{29} + \dots$$

$$(1-q)\, I_q(\Sigma_{2,3,7}) = \sum_{n=0}^{\infty} q^{-n(n+2)} (q^{n+1})_{n+1}$$

which have similar S-transformation with the 5-th/7-th order mock theta functions respectively.

Quantum modular form: WRT

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Asymptotic behaviors of the quantum invariants for hyperbolic manifolds are very different.

As an important example of hyperbolic manifolds, we pay attention to 4_1, whose N-colored Jones polynomial has the Habiro series

$$J_N(4_1;q) = \sum_{n=0}^{\infty}(-1)^n q^{-\frac{1}{2}n(n+1)}(q^{1-N})_n(q^{1+N})_n$$

Volume conjecture: geometry of quantum invariants

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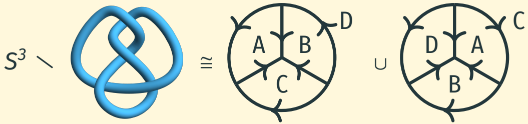
This expression gives $D(z) = \operatorname{Im} \operatorname{Li}_2(z) + \arg(1-z) \log |z|$.

$$\lim_{N \to \infty} \frac{2\pi}{N} \log \langle 4_1 \rangle_N = 2D(e^{\pi i/3}) = 2.02988\ldots$$

which is the hyperbolic volume $\operatorname{Vol}(S^3 \setminus 4_1)$ consisting of the two regular ideal tetrahedra.



The asymptotic expansion for hyperbolic knots looks like

$$\langle 4_1 \rangle_N \simeq N^{\frac{3}{2}} e^{2D(e^{\frac{\pi i}{3}})\frac{N}{2\pi}} \frac{1}{3^{1/4}} \left(1 + \frac{11}{36\sqrt{3}} \frac{\pi}{N} + \frac{697}{7776} \left(\frac{\pi}{N}\right)^2 + \dots \right)$$

There are several works on perturbative expansions [e.g., Dimofte–Gukov–Lenells–Zagier, Garoufalidis–Zagier, Fantini–Wheeler] using resurgence, state-integral, and so on.
"Quantum modularity" of hyperbolic manifolds needs further works.

We expect that a refined quantum invariant may help to get some insights in studying asymptotics.

DAHA & skein algebra

Refined quantum invariants & DAHA

Aganagic–Shakirov (2011) defined refined invariant for $T_{s,t}$, and pointed out a relationship with the Macdonald polynomials.

Cherednik constructed DAHA-Jones polynomial for $T_{s,t}$ which is related to Dunfield–Gukov–Rasmussen superpolynomial

$$P_R(K; a, q, t) = \sum_{i,j,k} a^i q^j t^k \dim \mathscr{H}(K)_R^{i,j,k}$$

$$\begin{tikzcd}
 & P_{\square}(K; a, q, t) \arrow[dl, "a = q^2"'] \arrow[dr, "t = -1"] & & \\
\begin{array}{c}\text{Khovanov}\\ Kh(K; q, t)\end{array} \arrow[dr, "t = -1"'] & & \begin{array}{c}\text{HOMFLY}\\ P(K; a, q)\end{array} \arrow[dl, "a = q^2"] \arrow[dr, "a = 1"] & \\
 & \begin{array}{c}\text{Jones}\\ J_2(K; q)\end{array} & & \begin{array}{c}\text{Alexander}\\ \Delta(K; q)\end{array}
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⚠️ Explicit forms of superpolynomial are known only for several knots e.g.

$$P_{S^N}(T_{2,3};a,q,t) = \sum_{k=0}^{N} q^{Nk} t^k \frac{(q)_N(-a/t)_k}{(q)_k(q)_{N-k}}$$

[Dunin-Barkowski–Mironov–Morozov–Sleptsov–Smirnov, Fuji–Gukov–Sulkowski]

A_1 DAHA is defined by $\qquad e = \frac{t+T}{t+t^{-1}}$

$$H_{q,t} = \left\langle T, X, Y \;\middle|\; \begin{array}{c} XTXT = 1, \quad YT^{-1}YT^{-1} = 1 \\ (T+t)(T-t^{-1}) = 0, \quad X^{-1}Y^{-1}XYT^{-2} = q^{-1} \end{array} \right\rangle, \qquad SH_{q,t} = eH_{q,t}e$$

whose polynomial representation is $\qquad (sf)(x) = f(x^{-1}), \quad (\eth f)(x) = f(qx)$

$$T \mapsto t^{-1}s + (t^{-1} - t)\frac{1}{x^2 - 1}(s - 1), \qquad X \mapsto x, \qquad Y \mapsto \eth s T$$

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This gives

$$(\mathrm{Y}+\mathrm{Y}^{-1})e \mapsto \left(\frac{tx - t^{-1}x^{-1}}{x - x^{-1}}\eth + \frac{t^{-1}x - tx^{-1}}{x - x^{-1}}\eth^{-1}\right)e$$

The eigen-polynomials are the A_1 Macdonald polynomials

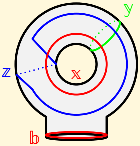
$$\left(\mathrm{Y}+\mathrm{Y}^{-1}\right) M_n(x;q,t) = \left(tq^n + t^{-1}q^{-n}\right) M_n(x;q,t)$$

which is explicitly written as

$$M_n(x;q,t) = \frac{(q^2;q^2)_n}{(t^2;q^2)_n}\sum_{k=0}^{n}\frac{(t^2;q^2)_{n-k}(t^2;q^2)_k}{(q^2;q^2)_{n-k}(q^2;q^2)_k}x^{n-2k}$$

Skein algebra on once-punctured torus $\mathrm{Sk}_A(\Sigma_{1,1})$ & A_1 DAHA $H_{q,t}$

The relationship between $\mathrm{Sk}_A(\Sigma_{1,1})$ and $H_{q,t}$ has been studied by Cherednik, Berest, Morton, Samuelson, … .



$\mathrm{Sk}_A(\Sigma_{1,1})$ is generated by simple closed curves $\mathbb{x}$, $\mathbb{y}$, $\mathbb{z}$ satisfying

$$A\mathbb{x}\mathbb{y} - A^{-1}\mathbb{y}\mathbb{x} = (A^2 - A^{-2})\mathbb{z}, \qquad \cdots \text{(cyclic in } \mathbb{x}, \mathbb{y}, \mathbb{z}) \cdots$$

$$\mathbb{b} = A\mathbb{x}\mathbb{y}\mathbb{z} - A^2\mathbb{x}^2 - A^{-2}\mathbb{y}^2 - A^2\mathbb{z}^2 + A^2 + A^{-2}$$

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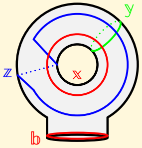

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Known is an isom. $\mathcal{A} : \mathrm{Sk}_{A=q^{-1/2}}(\Sigma_{1,1}) \to SH_{q,t}$ $\qquad \mathrm{ch}(X) = X + X^{-1}$

$$\mathbb{x} \mapsto \mathrm{ch}(\mathrm{X})\mathrm{e}, \quad \mathbb{y} \mapsto \mathrm{ch}(\mathrm{Y})\mathrm{e}, \quad \mathbb{z} \mapsto \mathrm{ch}(q^{\frac{1}{2}}\mathrm{XY})\mathrm{e}, \quad \mathbb{b} \mapsto -\mathrm{ch}(t^2q^{-1})\mathrm{e}$$

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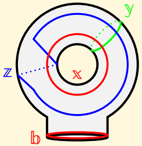

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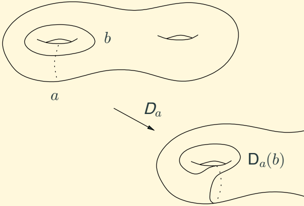
It is known that $SL_2(\mathbb{Z})$ acts on DAHA, whose generators are

$$\tau_R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : \begin{bmatrix} \mathrm{T} \\ \mathrm{Y} \\ \mathrm{X} \end{bmatrix} \mapsto \begin{bmatrix} \mathrm{T} \\ q^{1/2}\mathrm{XY} \\ \mathrm{X} \end{bmatrix}, \qquad \tau_L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} : \begin{bmatrix} \mathrm{T} \\ \mathrm{Y} \\ \mathrm{X} \end{bmatrix} \mapsto \begin{bmatrix} \mathrm{T} \\ \mathrm{Y} \\ q^{-1/2}\mathrm{YX} \end{bmatrix}$$

Automorphism $\mathrm{Aut}(H_{q,t})$ of DAHA & Dehn twists



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denote Dehn twist $\mathscr{D}_{\mathbb{x}}^{-1}$, $\mathscr{D}_{\mathbb{y}}$, respectively.

For the torus knot $T_{r,s}$, Cherednik's DAHA-Jones polynomial is

$$P_n(T_{r,s}; x, q, t) = \gamma_{r,s}(M_{n-1}(Y; q, t)) \cdot 1 \qquad \gamma_{r,s} = \begin{bmatrix} * & r \\ * & s \end{bmatrix} \in SL_2(\mathbb{Z})$$

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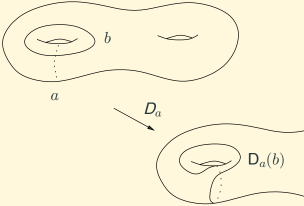

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Cherednik found that the A_N DAHA-polynomial gives the superpolynomial

$$\mathfrak{P}_{S^N}(A, q, t; T_{2,3}) = \sum_{i=1}^{N} (X_i Y_i)^2 X_i(1) \qquad \sum_{i=1}^{N} x_i^n \mapsto \frac{A^n - A^{-n}}{t^n - t^{-n}}$$

The relationship with $\mathrm{Sk}_A(\Sigma_{1,1})$ is a reason why DAHA is useful in quantum invariants. It is natural to expect a similar relationship for other DAHA.

As a rank 1 DAHA, we recall $C^\vee C_1$ DAHA $\qquad Y = T_1 T_0, \quad X = (T_1^\vee T_1)^{-1}$

$$H_{q,\mathbf{t}} = \left\langle T_0, T_1, T_0^\vee, T_1^\vee \;\middle|\; \begin{array}{c} T_1^\vee T_1 T_0 T_0^\vee = q^{-1/2} \\ (T_0 - t_0^{-1})(T_0 + t_0) = 0, \quad (T_1 - t_1^{-1})(T_1 + t_1) = 0 \\ (T_0^\vee - t_2^{-1})(T_0^\vee + t_2) = 0, \quad (T_1^\vee - t_3^{-1})(T_1^\vee + t_3) = 0 \end{array} \right\rangle$$

whose polynomial representation is

$$T_0 \mapsto t_0^{-1} s\eth - \frac{q^{-1}(t_0^{-1} - t_0)x^2 + q^{-1/2}(t_2^{-1} - t_2)x}{1 - q^{-1}x^2}(1 - s\eth), \qquad T_0^\vee \mapsto q^{-1/2} T_0^{-1} x$$

$$T_1 \mapsto t_1^{-1} s + \frac{(t_1^{-1} - t_1) + (t_3^{-1} - t_3)x}{x^2 - 1}(s - 1), \qquad T_1^\vee \mapsto x^{-1} T_1^{-1}$$

$C^\vee C_1$ DAHA

As a rank 1 DAHA, we recall $C^\vee C_1$ DAHA $\quad Y = T_1 T_0, \quad X = (T_1^\vee T_1)^{-1}$

$$H_{q,\mathbf{t}} = \left\langle T_0, T_1, T_0^\vee, T_1^\vee \;\middle|\; \begin{array}{c} T_1^\vee T_1 T_0 T_0^\vee = q^{-1/2} \\ (T_0 - t_0^{-1})(T_0 + t_0) = 0, \quad (T_1 - t_1^{-1})(T_1 + t_1) = 0 \\ (T_0^\vee - t_2^{-1})(T_0^\vee + t_2) = 0, \quad (T_1^\vee - t_3^{-1})(T_1^\vee + t_3) = 0 \end{array} \right\rangle$$

whose polynomial representation is

$$T_0 \mapsto t_0^{-1} s\eth - \frac{q^{-1}(t_0^{-1} - t_0)x^2 + q^{-1/2}(t_2^{-1} - t_2)x}{1 - q^{-1}x^2}(1 - s\eth), \qquad T_0^\vee \mapsto q^{-1/2} T_0^{-1} x$$

$$T_1 \mapsto t_1^{-1} s + \frac{(t_1^{-1} - t_1) + (t_3^{-1} - t_3)x}{x^2 - 1}(s - 1), \qquad T_1^\vee \mapsto x^{-1} T_1^{-1}$$

This gives

$$\left(Y + Y^{-1}\right) e \mapsto \left(A(x)(\eth - 1) + A(x^{-1})(\eth^{-1} - 1) + t_0 t_1 + (t_0 t_1)^{-1}\right) e$$

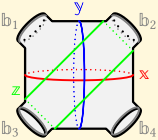
$$A(x) = t_0 t_1 \frac{\left(1 - \frac{x}{t_1 t_3}\right)\left(1 + \frac{t_3}{t_1}x\right)\left(1 - \frac{q^{1/2}}{t_0 t_2}x\right)\left(1 + \frac{q^{1/2} t_2}{t_0}x\right)}{(1 - x^2)(1 - qx^2)}$$

whose eigen-polynomial is the Askey–Wilson polynomial

$$\left(Y + Y^{-1}\right) A_m(x; q, \mathbf{t}) = \left(q^{-m} t_0 t_1 + q^m (t_0 t_1)^{-1}\right) A_m(x; q, \mathbf{t})$$

Skein algebra on 4-punctured sphere $\mathrm{Sk}_A(\Sigma_{0,4})$ & $C^\vee C_1$ DAHA $H_{q,\mathbf{t}}$

$\mathrm{Sk}_A(\Sigma_{0,4})$ is gen. by $\mathbb{x}$, $\mathbb{y}$, $\mathbb{z}$, satisfying



$$A^2\,\mathbb{x}\,\mathbb{y} - A^{-2}\,\mathbb{y}\,\mathbb{x} = \left(A^4 - A^{-4}\right)\,\mathbb{z} + \left(A^2 - A^{-2}\right)\left(\mathbb{b}_2\,\mathbb{b}_3 + \mathbb{b}_1\,\mathbb{b}_4\right)$$

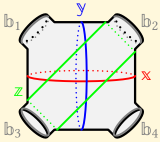
$$A^2\,\mathbb{y}\,\mathbb{z} - A^{-2}\,\mathbb{z}\,\mathbb{y} = \left(A^4 - A^{-4}\right)\,\mathbb{x} + \left(A^2 - A^{-2}\right)\left(\mathbb{b}_1\,\mathbb{b}_2 + \mathbb{b}_3\,\mathbb{b}_4\right)$$

$$A^2\,\mathbb{z}\,\mathbb{x} - A^{-2}\,\mathbb{x}\,\mathbb{z} = \left(A^4 - A^{-4}\right)\,\mathbb{y} + \left(A^2 - A^{-2}\right)\left(\mathbb{b}_1\,\mathbb{b}_3 + \mathbb{b}_2\,\mathbb{b}_4\right)$$

$$A^2\,\mathbb{x}\,\mathbb{y}\,\mathbb{z} = A^4\,\mathbb{x}^2 + A^{-4}\,\mathbb{y}^2 + A^4\,\mathbb{z}^2 + A^2\left(\mathbb{b}_1\,\mathbb{b}_2 + \mathbb{b}_3\,\mathbb{b}_4\right)\mathbb{x} + A^{-2}\left(\mathbb{b}_1\,\mathbb{b}_3 + \mathbb{b}_2\,\mathbb{b}_4\right)\mathbb{y}$$
$$+ A^2\left(\mathbb{b}_1\mathbb{b}_4 + \mathbb{b}_2\mathbb{b}_3\right)\mathbb{z} + \mathbb{b}_1^2 + \mathbb{b}_2^2 + \mathbb{b}_3^2 + \mathbb{b}_4^2 + \mathbb{b}_1\mathbb{b}_2\mathbb{b}_3\mathbb{b}_4 - \left(A^2 + A^{-2}\right)^2$$

Skein algebra on 4-punctured sphere $\mathrm{Sk}_A(\Sigma_{0,4})$ & $C^\vee C_1$ DAHA $H_{q,\mathbf{t}}$

$\mathrm{Sk}_A(\Sigma_{0,4})$ is gen. by $\mathbb{x}$, $\mathbb{y}$, $\mathbb{z}$, satisfying



$$A^2\,\mathbb{x}\,\mathbb{y} - A^{-2}\,\mathbb{y}\,\mathbb{x} = \left(A^4 - A^{-4}\right)\,\mathbb{z} + \left(A^2 - A^{-2}\right)\left(\mathbb{b}_2\,\mathbb{b}_3 + \mathbb{b}_1\,\mathbb{b}_4\right)$$

$$A^2\,\mathbb{y}\,\mathbb{z} - A^{-2}\,\mathbb{z}\,\mathbb{y} = \left(A^4 - A^{-4}\right)\,\mathbb{x} + \left(A^2 - A^{-2}\right)\left(\mathbb{b}_1\,\mathbb{b}_2 + \mathbb{b}_3\,\mathbb{b}_4\right)$$

$$A^2\,\mathbb{z}\,\mathbb{x} - A^{-2}\,\mathbb{x}\,\mathbb{z} = \left(A^4 - A^{-4}\right)\,\mathbb{y} + \left(A^2 - A^{-2}\right)\left(\mathbb{b}_1\,\mathbb{b}_3 + \mathbb{b}_2\,\mathbb{b}_4\right)$$

$$A^2\,\mathbb{x}\,\mathbb{y}\,\mathbb{z} = A^4\,\mathbb{x}^2 + A^{-4}\,\mathbb{y}^2 + A^4\,\mathbb{z}^2 + A^2\left(\mathbb{b}_1\,\mathbb{b}_2 + \mathbb{b}_3\,\mathbb{b}_4\right)\mathbb{x} + A^{-2}\left(\mathbb{b}_1\,\mathbb{b}_3 + \mathbb{b}_2\,\mathbb{b}_4\right)\mathbb{y}$$

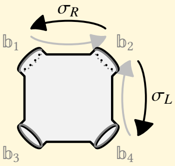
$$+ A^2\left(\mathbb{b}_1\mathbb{b}_4 + \mathbb{b}_2\mathbb{b}_3\right)\mathbb{z} + \mathbb{b}_1^2 + \mathbb{b}_2^2 + \mathbb{b}_3^2 + \mathbb{b}_4^2 + \mathbb{b}_1\mathbb{b}_2\mathbb{b}_3\mathbb{b}_4 - \left(A^2 + A^{-2}\right)^2$$

Oblomkov found $\mathcal{A} : \mathrm{Sk}_{A=q^{-1/4}}(\Sigma_{0,4}) \to SH_{q,\mathbf{t}}$ $\qquad \mathrm{ch}(X) = X + X^{-1}$

$$\mathbb{x} \mapsto \mathrm{ch}(X)e \qquad \mathbb{y} \mapsto \mathrm{ch}(Y)e \qquad \mathbb{z} \mapsto \mathrm{ch}(T_1 T_0^\vee)e$$

$$\mathrm{b}_1 \mapsto \mathrm{ch}(it_0)e \quad \mathrm{b}_2 \mapsto \mathrm{ch}(it_2)e \quad \mathrm{b}_3 \mapsto \mathrm{ch}(iq^{1/2}t_1)e \quad \mathrm{b}_4 \mapsto \mathrm{ch}(it_3)e$$

As before, $SL_2(\mathbb{Z})$ action on $H_{q,\mathbf{t}}$



$$\sigma_R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : \begin{bmatrix} T_0 \\ T_1 \\ T_0^\vee \\ T_1^\vee \end{bmatrix} \mapsto \begin{bmatrix} T_0 T_0^\vee T_0^{-1} \\ T_1 \\ T_0 \\ T_1^\vee \end{bmatrix}, \qquad \sigma_L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} : \begin{bmatrix} T_0 \\ T_1 \\ T_0^\vee \\ T_1^\vee \end{bmatrix} \mapsto \begin{bmatrix} T_0 \\ T_1 \\ T_1^\vee \\ T_1^{\vee -1} T_0^\vee T_1^\vee \end{bmatrix}$$

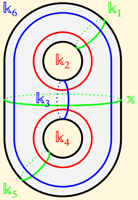
can be regarded as the (half) Dehn twists.

These give the DAHA polynomials for simple closed curves on $\Sigma_{0,4}$.

$\mathrm{Sk}_A(\Sigma_{2,0})$ is generated by $\Bbbk_1, \dots, \Bbbk_6$ with many relations.

The mapping class group is generated by the Dehn twists $\mathscr{D}_i$ along $\Bbbk_i$

$$\mathscr{D}_{i,j,\dots,k} = \mathscr{D}_i \mathscr{D}_j \dots \mathscr{D}_k$$



$$\mathrm{MCG}(\Sigma_{2,0}) = \left\langle \mathscr{D}_1, \dots, \mathscr{D}_5 \;\middle|\; \begin{array}{c} \mathscr{D}_{i,i+1,i} = \mathscr{D}_{i+1,i,i+1} \text{ for } 1 \le i \le 4 \\ \mathscr{D}_{i,j} = \mathscr{D}_{j,i} \text{ for } |i-j| > 1 \\ \left(\mathscr{D}_{1,2,3,4,5}\right)^6 = 1 \\ \left(\mathscr{D}_{5,4,3,2,1,1,2,3,4,5}\right)^2 = 1 \end{array} \right\rangle$$

Skein algebra on genus 2 surface $\mathrm{Sk}_A(\Sigma_{2,0})$

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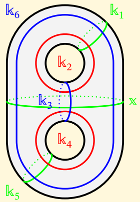
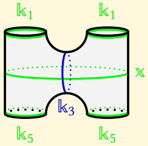


$$\mathrm{MCG}(\Sigma_{2,0}) = \left\langle \mathscr{D}_1, \dots, \mathscr{D}_5 \;\middle|\; \begin{array}{c} \mathscr{D}_{i,i+1,i} = \mathscr{D}_{i+1,i,i+1} \text{ for } 1 \le i \le 4 \\ \mathscr{D}_{i,j} = \mathscr{D}_{j,i} \text{ for } |i-j| > 1 \\ \left(\mathscr{D}_{1,2,3,4,5}\right)^6 = 1 \\ \left(\mathscr{D}_{5,4,3,2,1,1,2,3,4,5}\right)^2 = 1 \end{array} \right\rangle$$

We need to construct a polynomial representation of generators $\Bbbk_a$ by use of generalized $C^\vee C_1$ DAHA.

⚠️ Arthamonov–Shakirov initiated studies on DAHA for $\mathrm{Sk}_A(\Sigma_{2,0})$ based on A_1 DAHA. We took a different approach based on $C^\vee C_1$ DAHA, which contributes to $\mathrm{Sk}_A(\Sigma_{0,4})$.



With $C^\vee C_1$ DAHA T_a, we have the Iwahori–Hecke operators $\quad (\eth_0 g)(x_0) = g(q^{1/2}x_0)$

$$\mathrm{T}_0 \mapsto \mathrm{i}\,\frac{x}{q^{\frac{1}{2}}-x}\left(-\frac{q^{\frac{1}{2}}+x\,x_0^2}{x\,x_0}\,\mathrm{s}\,\eth + x_0 + x_0^{-1}\right); \qquad G_n(x_0;x) = \frac{-x_0^{-n}}{1-x_0^2}\,\eth_0 + \frac{x_0^n(q^{\frac{1}{2}}x+x_0^2)(q^{\frac{1}{2}}+x\,x_0^2)}{q^{\frac{1}{2}}\,x\left(1-x_0^2\right)}\,\eth_0^{-1}$$

$$\mathrm{T}_1 \mapsto \mathrm{i}\left(\frac{1+q^{\frac{1}{2}}x}{q^{\frac{1}{2}}(1-x^2)}\,\frac{q^{\frac{1}{2}}x+x_1^2}{x_1}\,(\mathrm{s}-1) - q^{\frac{1}{2}}x_1^{-1}\right); \qquad K_n(x_0;x) = \frac{-x_0^{-n}}{1-x_0^2}\,\eth_0 + \frac{x_0^n(q^{\frac{1}{2}}x+x_0^2)(q^{\frac{3}{2}}x+x_0^2)}{q\,x\left(1-x_0^2\right)}\,\eth_0^{-1}$$

$$\mathrm{U}_0 \mapsto \frac{q^{-\frac{1}{4}}x}{q^{\frac{1}{2}}-x}\,K_0(x_0;x^{-1})\,\mathrm{s}\,\eth - \frac{q^{-\frac{1}{4}}x}{q^{\frac{1}{2}}-x}\,G_0(x_0;x)$$

$$\mathrm{U}_1 \mapsto -\frac{x(1+q^{\frac{1}{2}}x)}{q^{\frac{1}{4}}(1-x^2)}\,K_0(x_1;x)\,(\mathrm{s}-1) + \frac{q^{\frac{1}{4}}}{1-q^{\frac{1}{2}}x}\left(G_0(x_1;x) - q^{\frac{1}{2}}x\,K_0(x_1;x)\right)$$

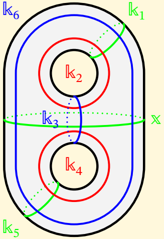
Then the simple closed curves $\Bbbk_a$ are

$$\Bbbk_1 \mapsto \mathrm{ch}\,(\mathrm{i}\,\mathrm{T}_0)\,e \qquad \Bbbk_2 \mapsto \mathrm{ch}\,(\mathrm{i}\,\mathrm{U}_0)\,e$$

$$\Bbbk_3 \mapsto \mathrm{ch}(\mathrm{T}_1\mathrm{T}_0)\,e = \mathrm{ch}(\mathrm{T}_0\mathrm{T}_1)\,e, \qquad \Bbbk_4 \mapsto \mathrm{ch}\left(\mathrm{i}\,q^{-\frac{1}{2}}\mathrm{U}_1\right)e$$

$$\Bbbk_5 \mapsto \mathrm{ch}\left(\mathrm{i}\,q^{-\frac{1}{2}}\mathrm{T}_1\right)e \qquad \Bbbk_6 \mapsto \mathrm{ch}\,(\mathrm{U}_1\mathrm{U}_0)\,e$$

$$\mathbb{x} \mapsto \mathrm{ch}(x)e$$



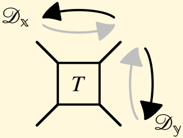
Iwahori–Hecke relations

We have

$$
\begin{gathered}
\left(\mathrm{T}_0 + \mathrm{i}\, x_0\right)\left(\mathrm{T}_0 + \mathrm{i}\, x_0^{-1}\right) = 0,\\
\left(\mathrm{T}_1 + \mathrm{i}\, q^{-1/2} x_1\right)\left(\mathrm{T}_1 + \mathrm{i}\, q^{1/2} x_1^{-1}\right) = 0,\\
\left(\mathrm{U}_0 + \frac{q^{-1/4}x}{q^{1/2} - x}\left(G_0(x_0; x) - K_0(x_0; \tfrac{1}{x})\right)\right)\left(\mathrm{U}_0 + \frac{q^{-1/4}}{q^{1/2} - x}\left(x\, G_0(x_0; x) - q^{1/2} K_0(x_0; \tfrac{x}{q})\right)\right) = 0,\\
\left(\mathrm{U}_1 - \frac{q^{1/4}}{1 - q^{1/2}x}\left(G_0(x_1; x) - q^{1/2} x\, K_0(x_1; x)\right)\right)\left(\mathrm{U}_1 - \frac{q^{3/4}}{q^{3/2} - x}\left(G_0(x_1; \tfrac{x}{q}) - K_0(x_1; \tfrac{q}{x})\right)\right) = 0.
\end{gathered}
$$

Conway rational tangle & automorphism of gen. DAHA

Conway rational tangle is an isomorphism $\mathbb{Q} \to$ tangle, which is given using a continued fraction $\frac{p}{q} \in \mathbb{Q}$



$$\frac{p}{q} = [a_1, a_2, \ldots, a_n] = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}$$

Here a_{odd} and a_{even} are the numbers of half Dehn twists $\mathscr{D}_{\mathbb{x}}$ and $\mathscr{D}_{\mathbb{y}}$ respectively.

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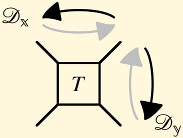

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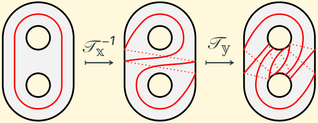
Here a_{odd} and a_{even} are the numbers of half Dehn twists $\mathscr{D}_{\mathbb{x}}$ and $\mathscr{D}_{\mathbb{y}}$ respectively. In our generalized DAHA side, the full Dehn twists are automorphisms

$$\mathscr{T}_{\mathbb{x}} : \begin{bmatrix} T_0 \\ T_1 \\ X \\ U_0 \\ U_1 \end{bmatrix} \mapsto \begin{bmatrix} XT_0X^{-1} \\ T_1 \\ X \\ XU_0X^{-1} \\ U_1 \end{bmatrix}, \quad \mathscr{T}_{\mathbb{y}} : \begin{bmatrix} T_0 \\ T_1 \\ X \\ U_0 \\ U_1 \end{bmatrix} \mapsto \begin{bmatrix} T_0 \\ T_1 \\ (T_0T_1)^{-1}XT_1T_0 \\ q^{-1/4}(T_0T_1)^{-1}U_0 \\ q^{1/4}U_1T_0T_1 \end{bmatrix}$$

- $4_1 \sim \frac{5}{2} = 2 + \frac{1}{2} \leftrightarrow \widetilde{\mathbb{y}}_{5/2} = \left(\mathscr{D}_{\mathbb{y}}\mathscr{D}_{\mathbb{x}}^{-1}\right)(\widetilde{\mathbb{y}}),$



$$\mathcal{A}(\widetilde{\mathbb{y}}_{5/2}) = \mathrm{ch}\left(U_1T_0T_1\left(T_0^{\vee}T_1T_0\right)^{-1}T_0^{-1}U_0T_1\left(T_0^{\vee}T_1T_0\right)\right)e$$

- $5_2 \sim \frac{7}{2} = 4 + \frac{1}{-2} \leftrightarrow \widetilde{\mathbb{y}}_{7/2} = \left(\mathscr{D}_{\mathbb{y}}^{-1}\mathscr{D}_{\mathbb{x}}^{-2}\right)(\widetilde{\mathbb{y}}),$

$$\mathcal{A}(\widetilde{\mathbb{y}}_{7/2}) = \mathrm{ch}\left(U_1T_1^{-1}T_0^{-1}\left(T_1T_0T_1^{\vee}T_0^{-1}\right)T_1T_0T_1^{\vee}T_1U_0\left(T_1T_0T_1^{\vee}T_0^{-1}\right)^{-2}\right)e$$

We have checked that the Jones polynomial recover from

$$\mathrm{Const}_{\mathfrak{d}^0}(\mathcal{A}(\widetilde{\mathbb{y}}_{5/2}))(1)\Big|_{x_0=x_1=-x=q^{\frac{1}{2}}} = \frac{q^{-2}-q^{-1}+1-q+q^2}{(1-q)(1-q^2)}$$

$$\mathrm{Const}_{\mathfrak{d}^0}(\mathcal{A}(\widetilde{\mathbb{y}}_{7/2}))(1)\Big|_{x_0=x_1=-x=q^{\frac{1}{2}}} = \frac{q(1-q+2q^2-q^3+q^4-q^5)}{(1-q)(1-q^2)}$$

Concluding Remarks

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We have emphasized that the quantum invariants of knots & 3-manifolds are interesting both from geometry & number theory.

- volume conjecture
- quantum modularity

In this talk, we have discussed the topological aspects of DAHA using the skein algebra on surfaces. We are at a initial stage in studying DAHA polynomials. We want to work

- relationship with the refined invariants
- DAHA for $\mathrm{Sk}_A(\Sigma_{g,n})$
- quantum modularity

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🙏 感謝 🙏

😊 Thanks to Suzuki-san and Okazaki-san for organizing Workshop.