PP-wave Holography and Entropy Bounds

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B. Du, MH, JHEP 03 (2021) 246, arXiv:2101.07484;
B. Du, MH, JHEP 08 (2021) 006, arXiv:2104.12502;
MH, Phys.Lett.B 846 (2023) 138231, arXiv:2307.01847;
MH, arXiv:2511.08213.

Outline of the Talk

1. PP-wave Holography

   (a) Background and introduction

   (b) A probability interpretation of BMN two-point functions

2. Entropy Bounds

   (a) Entropy of BMN strings

   (b) Thermal entropy in Calabi-Yau quantum systems

   (c) A possible cosmological application

- The AdS/CFT correspondence (Maldacena, 1997) is a deep idea which relates two seemingly totally different theories, namely type IIB string theory or supergravity on $AdS_5 \times S^5$ background and the $\mathcal{N} = 4$ $SU(N)$ super-Yang-Mills theory.

- Although the correspondence has found flourishing applications in many topics, the precise quantitative tests of the holographic dictionary are mostly restricted to supersymmetry protected quantities in the supergravity approximation, such as the spectrum and correlation functions of BPS operators.

- Without an alternative effective method to handle string theory in the deeply stringy regime, a common perspective is to simply take the super-Yang-Mills theory as a non-perturbative definition of AdS string theory at any coupling and energy scale, assumed to be valid unless otherwise convincingly explicitly contradicted.

- A particularly interesting avenue for progress in the precise tests of the holographic correspondence in the stringy regime is to take a Penrose limit of the type IIB $AdS_5 \times S^5$ background.

- The geometry becomes a pp-wave or plane wave background (Blau, Figueroa-O'Farrill, Hull, Papadopoulos, 2001), with also maximal supersymmetry

$$ds^2 = -4dx^+dx^- - \mu^2(\vec{r}^{\,2} + \vec{y}^{\,2})(dx^+)^2 + d\vec{r}^{\,2} + d\vec{y}^{\,2}, \tag{1}$$

where x^+, x^- are light cone coordinates, $\vec{r}, \vec{y}$ are 4-vectors, and the parameter μ measures the spacetime curvature as well as the Ramond-Ramond flux $F_{+1234} = F_{+5678} \sim \mu$. ($\mu = 0$ is simply the Minkowski space in light cone coordinates)

- The free string spectrum can be obtained in the light cone gauge using Green-Schwarz formalism similar to the flat space. (Unlike AdS space)

- In the groundbreaking paper, Berenstein, Maldacena and Nastase (BMN 2002) proposed a type of near-BPS operators, which correspond to the type IIB closed strings on the pp-wave background. The free string spectrum is correctly reproduced by perturbative gauge interactions as the planar conformal dimensions of BMN operators.

- The BMN scaling limit with large R-charge $J \sim \sqrt{N} \sim \infty$ appears to be the right Goldilocks limit in this situation.
  A smaller R-charge would not provide finite string interactions in the strict $N \sim \infty$ limit,
  while a larger R-charge may blow up the strings into D-branes, known as giant gravitons.

- This appears to be a promising ground for quantitative explorations of the holographic duality in stringy regimes, as the dual theories on both sides can be either free or weakly coupled.
  Caveat: spacetime highly (even infinitely) curved!

More Physical Motivation

- A main goal of the theories of quantum gravity is to understand the physics in the regime beyond the reach of classical gravity, e.g. in the highly curved spacetime region near the black hole singularity.

- For AdS_d space with $d > 2$, the scalar curvature is negative $R = -\frac{d(d-1)}{r^2}$ where r is the radius of the AdS space, so the spacetime is highly curved if the radius r is very small (compared to the string or Planck length).

- For the pp-wave background, the scalar curvature actually vanishes, while the non-vanishing component of the Ricci curvature is proportional to μ^2. We will focus on the $\mu \sim \infty$ or infinite curvature limit.

- Certainly the effective action of classical gravity completely breaks down in this case, but this is not necessarily a bad situation as we can instead probe the fundamental nature of spacetime in a stringy regime.
  The classical geometry blows up and is very singular, but the quantum theory is completely finite and consistent.

- The Penrose limit provides a new twist to the holography story. In the celebrated AdS/CFT holographic dictionary Witten:1998, the CFT lives at the boundary of a bulk AdS space and its local operators couple to the boundary configurations of the AdS bulk fields.

- However, although the pp-wave background comes from a Penrose limit of the AdS space, the geometry is rather different. As such, it is not clear how to directly apply the standard AdS holographic dictionary, particularly in the situations with finite string interactions.

- Our approach in some previous papers Huang:2002a, Huang:2002b, Huang:2010, Huang:2019a, Huang:2019b is to consider another corner of the parameter space in the BMN limit, focusing on the free gauge theory. In this case, the string theory side becomes infinitely curved $\mu \sim \infty$, and strings are effectively infinitely long and tensionless, but can still have finite string interactions. Some non-planar contributions to BMN correlators were first computed in e.g. Constable, Freedman, Headrick, Minwalla, Motl, Postnikov, Skiba, 2002.

- Most interestingly, since the string spectrum is completely degenerate, the tensionless string can jump from one excited state to another without energy cost through a quantum unitary transition. It turns out that in this case the effective string coupling constant should be identified with a finite genus counting parameter $g := \frac{J^2}{N}$.

- Since the full fledged holographic dictionary is no longer available in the pp-wave background, our pragmatic approach is to try to compute the physical quantities on both sides of the correspondence and find potential non-trivial agreements.

- In this sense, a mismatch with naive expectation is not necessarily a contradiction of the holographic principle. Instead, one should focus on finding aspects where the calculations from both side do match, and try to give physical derivations or proofs of such mathematical coincidence.

- Besides the free string spectrum originally considered in BMN 2002 (0th entry), there are some more tests of the pp-wave holography. Three entries (we will refer to them as the 1st, 2nd, 3rd entries) of "pp-wave holographic dictionary":

  1. The free planar three-point functions of BMN operators correspond to the Green-Schwarz light cone string field cubic vertex in the infinite curvature limit. Spradin and Volovich 2002; Huang:2002a

  2. In the papers Huang:2002b, Huang:2010, we further proposed a factorization formula, where the free (higher genus) BMN correlators are holographically related to string (loop) diagram calculations by pasting together the cubic string vertices without propagator.

  3. More recently, we propose a probability interpretation of the BMN two-point functions Huang:2019a, with indirect tests through mostly the non-negativity of BMN two-point functions. This entry (focus of this talk) seems most interesting, though the evidence is indirect.

Some Notations

- The BMN vacuum operator is simply proportional Tr (Z^J) where Z is a complex scalar in the $\mathcal{N} = 4$ $SU(N)$ super-Yang-Mills theory. One can insert the four remaining real scalars into the trace with phases, corresponding to string modes in four of the eight transverse dimensions.

- The BMN operators are then denoted as $O^J_{(m_1, m_2, \cdots, m_k)}$, where the positive and negative integer modes represent the left and right moving stringy excited modes, while the zero modes are supergravity modes representing discretized momenta in the corresponding traverse direction.

- Due to the closed string level matching condition

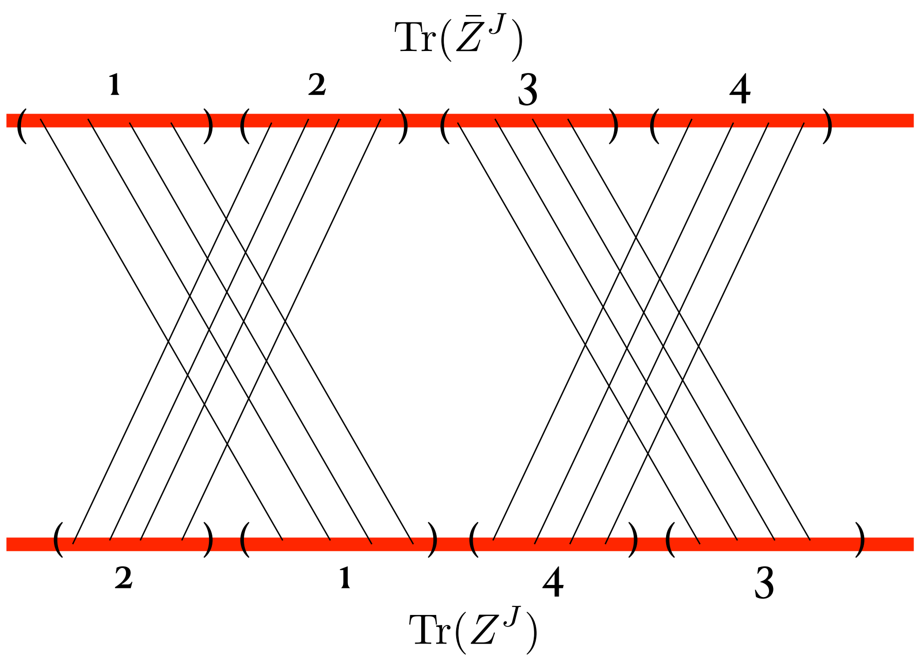
$$\sum_i m_i = 0, \tag{2}$$

the excited stringy states have at least two string modes.

- The BMN operators are properly normalized to be orthonormal at planar level, and the genus h two-point functions are proportional to g^{2h} as

$$\begin{aligned} \langle \bar{O}^J_{(m_1,m_2,\cdots,m_k)} O^J_{(n_1,n_2,\cdots,n_k)} \rangle_0 &= \delta_{m_1,n_1} \cdots \delta_{m_k,n_k}, \\ \langle \bar{O}^J_{(m_1,m_2,\cdots,m_k)} O^J_{(n_1,n_2,\cdots,n_k)} \rangle_h &\sim g^{2h}. \end{aligned} \tag{3}$$

- An example of torus (genus one) diagram



The Probability Interpretation

- The BMN two-point functions are real and symmetric, and there is a nice normalization relation summing over one set of mode numbers

$$\sum_{\sum_{i=1}^{k} n_k = 0} \langle \bar{O}^J_{(m_1,m_2,\cdots,m_k)} O^J_{(n_1,n_2,\cdots,n_k)} \rangle_h = \frac{g^{2h}}{2^{2h}(2h+1)!}. \tag{4}$$

- We define a matrix element, summing up all genus contributions with a proper normalization by the all-genera formula of vacuum correlator

$$p_{(m_1,m_2,\cdots,m_k),(n_1,n_2,\cdots,n_k)} = \frac{g}{2\sinh(\frac{g}{2})} \sum_{h=0}^{\infty} \langle \bar{O}^J_{(m_1,m_2,\cdots,m_k)} O^J_{(n_1,n_2,\cdots,n_k)} \rangle_h, \tag{5}$$

so that it looks like a probability distribution

$$\sum_{\sum_{i=1}^{k} n_k = 0} p_{(m_1,m_2,\cdots,m_k),(n_1,n_2,\cdots,n_k)} = 1. \tag{6}$$

- To interpret the matrix elements as a probability distribution, they need to be non-negative.

- For the case of two string modes, the non-negativity at any genus can be easily proven since the two string modes are opposite numbers Huang:2019a.

- For the case of four string modes, it turns out that the genus one two-point functions can be negative. There seems to be a rule forbidding the "crowdedness" of string modes, that we can not holographically use up all four remaining scalars to fully occupy the four transverse dimensions with $SO(4)$ rotational symmetry unbroken by the Ramond-Ramond flux in the pp-wave background.

- For the case of three string modes, we are not aware of a simple analytic proof of the non-negativity. Instead, we perform the detailed calculations and explicitly verify the non-negativity up to genus two.

A Novel Entry

- The results suggest a novel entry (the 3rd entry) of the pp-wave holographic dictionary

$$p_{(m_1,\cdots,m_k),(n_1,\cdots,n_k)} = |\langle m_1,\cdots,m_k|\widehat{U}(g)|n_1,\cdots,n_k\rangle|^2, \quad k=2,3, \tag{7}$$

  where the operator $\widehat{U}(g)$ describes the quantum unitary transition between the degenerate tensionless strings. The BMN single string states form a complete orthonormal basis of the Hilbert space under such finite string interactions

$$\sum_{\sum_{i=1}^{k} n_k = 0} |n_1, n_2, \cdots, n_k\rangle\langle n_1, n_2, \cdots, n_k| = I. \tag{8}$$

- Of course, the probability interpretation only requires the matrix element p is non-negative. Here we make the stronger conjecture that the BMN two-point functions with three string modes are always non-negative separately at each genus.

- This seemingly simple equation (7) has eluded research on the topic for many years, as the physical picture of string dynamics in the infinitely curved pp-wave background turns out to be drastically different from those familiar in flat spacetime or AdS space with large radius.

- In particular, there is no finite physical process of multiple particles or strings scattering to and from asymptotic region of spacetime, as the higher point functions always vanish in the strict $N \sim \infty$ limit. (Of course, they are still very useful, as in the 1st, 2nd entries, since infinitely many of them may combine to make a finite contribution.)

- Instead, the tensionless string directly jumps from one excited state to another through a quantum unitary transition, as in a S-matrix. This is an emergent time.

- Since the only finite BMN two-point functions are always real and symmetric, unitarity of string interactions rules out the naive possibility that they are directly identified with quantum transition amplitudes on the string theory side Huang:2019a. So we arrive at the otherwise seemingly most natural conjecture (7).

Torus Two-point Functions

- The formula with k string modes

$$
\begin{aligned}
&\langle \bar{O}^J_{(m_1,m_2,\cdots,m_k)} O^J_{(n_1,n_2,\cdots,n_k)} \rangle_{\text{torus}} \\
&= \frac{g^2}{4} \int_0^1 dx_1 dx_2 dx_3 dx_4 \delta(x_1+x_2+x_3+x_4-1) \times \\
&\prod_{i=1}^{k} \Big(\int_0^{x_1} + e^{2\pi i n_i (x_3+x_4)} \int_{x_1}^{x_1+x_2} + e^{2\pi i n_i (x_4-x_2)} \int_{x_1+x_2}^{1-x_4} + e^{-2\pi i n_i (x_2+x_3)} \int_{1-x_4}^{1} \Big) dy_i e^{2\pi i (n_i-m_i) y_i} \\
&= g^2 \int_0^1 dx_1 dx_2 dx_3 dx_4 \delta(x_1+x_2+x_3+x_4-1) \int_0^{x_1} dy_k e^{2\pi i (n_k-m_k) y_k} \times \\
&\prod_{i=1}^{k-1} \Big(\int_0^{x_1} + e^{2\pi i n_i (x_3+x_4)} \int_{x_1}^{x_1+x_2} + e^{2\pi i n_i (x_4-x_2)} \int_{x_1+x_2}^{1-x_4} + e^{-2\pi i n_i (x_2+x_3)} \int_{1-x_4}^{1} \Big) dy_i e^{2\pi i (n_i-m_i) y_i}.
\end{aligned}
$$

- Reality: We take the complex conjugate in the first formula, and change the integration variables $y_i \to 1-y_i, i=1,2,\cdots,k$ and $x_1,x_2,x_3,x_4 \to x_4,x_3,x_2,x_1$. After a simple calculation, also using the closed string level matching condition, one can check the formula remains the same. So the torus two-point function is purely real and symmetric.

Two String Modes

$$\langle \bar{O}^J_{(m_1,m_2)} O^J_{(n_1,n_2)} \rangle_{\text{torus}}$$
$$= \frac{g^2}{4} \int_0^1 dx_1 dx_2 dx_3 dx_4 \delta(x_1 + x_2 + x_3 + x_4 - 1) \times$$
$$\prod_{i=1}^{2} \left(\int_0^{x_1} + e^{2\pi i n_i (x_3+x_4)} \int_{x_1}^{x_1+x_2} + e^{2\pi i n_i (x_4 - x_2)} \int_{x_1+x_2}^{1-x_4} + e^{-2\pi i n_i (x_2+x_3)} \int_{1-x_4}^{1} \right) dy_i e^{2\pi i (n_i - m_i) y_i}.$$

Since $m_1 + m_2 = n_1 + n_2 = 0$. This is apparently non-negative.

- Higher genus: This works similarly.
  Two string modes: The non-negativity is easily proven.
  Any string modes: Flipping the diagrams vertically and horizontally, we can show the formula is always real and symmetric.

Some Standard Integrals

- We will use some standard integrals, which is defined by

$$I(u_1, u_2, \cdots, u_r) \equiv \int_0^1 dx_1 \cdots dx_r \delta(x_1 + \cdots + x_r - 1) e^{2\pi i(u_1 x_1 + \cdots u_r x_r)}.$$

  It is clear that the integral is unchanged if we add an integer to all the arguments. If some of the u_i's are identical, one uses the following notation

$$I_{(a_1, \cdots, a_r)}(u_1, u_2, \cdots, u_r) \equiv I(u_1, \cdots, u_1, u_2, \cdots, u_2, \cdots, u_r, \cdots, u_r),$$

  where a_i's are integers representing the numbers of the u_i's in the right hand side, and for $a_i = 0$ we can just eliminate the corresponding argument.

- The integral can be calculated by some recursion relations. A useful special case is when all arguments are degenerate at an integer

$$I_{n+1}(0) = \int_0^1 dx_1 \cdots dx_{n+1} \delta(x_1 + \cdots + x_{n+1} - 1) = \frac{1}{n!},$$

  which is simply the volume of the standard n-dimensional simplex

Three String Modes

- The BMN two-point functions can be calculated in terms of the standard integrals.
  Generic case: no degeneracy of string mode numbers in the standard integrals, i.e. none of $m_i, n_i, m_i \pm n_j$'s is zero.

- The genus one formula for the generic case

$$\langle \bar{O}^J_{(m_1,m_2,m_3)} O^J_{(n_1,n_2,n_3)} \rangle_{\text{torus}} = \frac{g^2}{32\pi^4} \frac{\sum_{i=1}^3 (m_i - n_i)^2}{\prod_{i=1}^3 (m_i - n_i)^2}, \tag{9}$$

  which is of course manifestly positive.

- We check also the numerous degenerate cases where some of $m_i, n_i, m_i \pm n_j$'s vanishes. Most cases also have manifestly non-negative results. However, there are several somewhat complicated degenerate cases. We perform a careful analysis to provide a complete proof of non-negativity at genus one for any mode numbers.

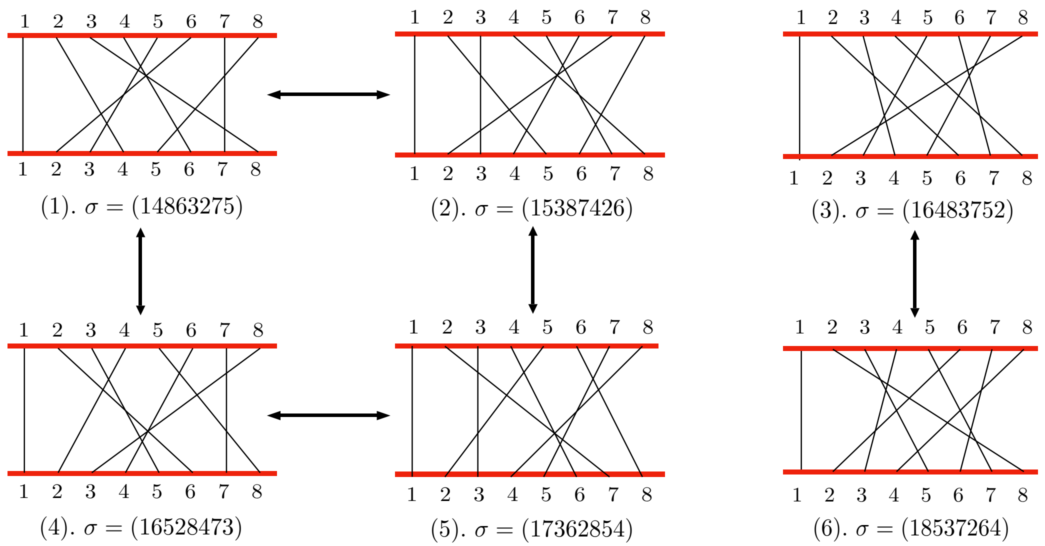
- Three string modes, Genus two: there are 21 diagrams. The calculations are quite complicated. After some very lengthy calculations, we finally obtain the result in terms of standard integrals, which is organized into four parts and too long to write down here.

- It is important to go through the laborious calculations, in order to ensure the previously observed non-negativity at genus one is not just a lucky coincidence, but more likely a manifestation of the deep mathematical structures of the underlying holographic duality.

- These 21 permutations are divided into 4 groups according to cyclicity

  1. (14732865), (17548362), (18643725), (14875326), (15837642), (18472653), (15428736), (17625843),
  2. (15387426), (15842763), (16528473), (17362854), (17438625), (14863275), (16483752), (18537264),
  3. (14325876), (14765832), (18365472), (18725436),
  4. (16385274).

- Symmetries: an example



- We check numerically for all cases with mode numbers $\max(|m_i|, |n_i|) \leq 30$. Furthermore, we compute the result for more than a million random sets of mode numbers in larger range. In all tests we have not found any negative result.

- For the case of generic mode numbers with large absolute value, the dominant term can be calculated. It is actually proportional to the genus one formula and is manifestly positive.

Some Discussions

- The probability interpretation of two-point function implies the genus expansion is convergent. In this sense, the holographic higher genus calculations are not asymptotic perturbative expansions as familiar in most examples of quantum theories.

- If no new non-perturbative effect is discovered in the future, then perhaps we have luckily found a rare example of perturbatively complete string theory.

- Thus, if our proposal of the entry of pp-wave holographic dictionary (7) is correct, to our knowledge, it would not only provide first examples of systematic calculations of (the norms of) the higher genus critical superstring amplitudes, but may also in principle gives exact complete results for any string coupling, due to the convergence of genus expansion.

- There is no technical obstruction for our calculations on the free gauge theory side at any genus, however the available tools on the string theory side are very limited. Much progress for the calculations of higher genus critical superstring amplitudes focused on using the RNS formalism in flat space, and is already quite difficult at genus two.

- Our conjecture (7) gives the norms of certain critical superstring amplitudes including all genus contributions. Of course, the string amplitudes are complex, and consist of the norms and phase angles. As discussed in Huang:2019a, unitarity can in principle determine a large part of the phase angles, but not completely.

- Our studies thus provide a long term motivation for future research to develop techniques that can deal with string theory on highly curved background, with flux, and including highly excited stringy states, for the purposes of a direct verification of the conjecture (7) as well as the complete determination of string amplitudes including the phase angles.
  For the moment, our verifications of non-negativity on the gauge theory side provide indirect non-trivial evidence of the conjecture (7).

- Now that we are more confident in the validity of the conjectured non-negativity for three string modes, it would be certainly better to search for a universal analytic proof, at genus two and further at any higher genus. Volumes of some geometric objects?

- It is widely believed that strings may not be the right fundamental degrees of freedom to formulate the still mysterious M-theory, since there are non-perturbative objects like D-branes and M-branes.

- On the contrary, if our conjecture (7) is correct, it seems that in our very special setting with infinite spacetime curvature and Ramond-Ramond flux, the tensionless closed strings do provide the proper complete degrees of freedom for physics at any coupling constant.

- In other words, we conjecture that in this (highly unrealistic) situation, string theory is really a theory of just "strings" .

Entropy of BMN strings

- Entropy has played a significant role in recent developments in our understandings of holography and quantum gravity. Since the seminal proposal of holographic entanglement entropy in Ryu and Takayanagi, 2006, there have been many generalizations and applications.

- In particular, a generalized version of the entropy may follow the Page curve of the black hole entropy during the Hawking radiation process, as reviewed in Almheiri et al, 2020. These significant progress appear to be getting close toward a consensus on the resolution of the famous black hole information paradox.

- In classical physics the entropy is only defined up to a constant (see e.g. some of Witten's papers). In quantum physics, the entropy can be thought of as the logarithm of the effective dimension of the Hilbert space (no more than the actual dimension, could be infinite) (we set $k_B = 1$).

- For convenience we consider the case of two string modes and denote $p_{m,n} \equiv p_{(-m,m),(-n,n)}$. Our probability interpretation provides a natural definition for the entropy of BMN strings

$$S_m(g) = -\sum_{n=-\infty}^{\infty} p_{m,n} \log(p_{m,n}).$$

- Some discussions.
  Similar to Shannon entropy in information theory.
  The initial BMN pure state has zero entropy. The quantum evolution is unitary.
  An observer makes a measurement without revealing the result, we get a mixed state with finite entropy. Also known as decoherence.
  The BMN basis is the preferred basis for the measurement.
  This can be also understood as the entanglement entropy between the observer and the BMN string.

- We will derive an upper bound for the entropy.

- A bound for the correlator.

  For genus h, the field theory calculations of the two-point amplitude $\langle \bar{O}^J_{-m,m} O^J_{-n,n} \rangle_h$ consist of $\frac{(4h-1)!!}{2h+1}$ cyclically inequivalent diagrams of dividing the long string into $4h$ segments.

$$\begin{aligned} & \langle \bar{O}^J_{-m,m} O^J_{-n,n} \rangle_h \\ \leq \;& \frac{(4h-1)!!}{2h+1} \frac{4hg^{2h}}{\pi^2(m-n)^2} \int_0^1 dx_1 \cdots dx_{4h} \delta(\sum_{i=1}^{4h} x_i - 1) \\ = \;& \frac{16h^2 g^{2h}}{2^{2h}(2h+1)!\pi^2(m-n)^2}. \end{aligned}$$

  The strength of BMN string interactions are bounded by an inverse square law.

- Summing over all genera, we have an estimate of the probability matrix element as

$$p_{m,n} \leq \frac{f(g)}{\pi^2(m-n)^2},$$

  where $f(g) := \frac{2g}{\sinh(g/2)}[\frac{g^2+4}{2g}\sinh(\frac{g}{2}) - \cosh(\frac{g}{2})] \sim g^2$.

- First notice for $0 < p < 1$, the function $-p\log(p)$ achieved maximum at $p = e^{-1}$, and it is monotonic in $p \in (0, e^{-1})$. We can choose an integer
$$n_0 \geq \max(\frac{\sqrt{e \cdot f(g)}}{\pi}, 2),$$
and evaluate the sum in 3 parts for $n \leq m - n_0$, $m - n_0 < n < m + n_0$, and $n \geq m + n_0$.

- The two parts that extends to $\pm\infty$ are symmetric with the same contributions, and in the middle part the entropy is maximal with a uniformly distributed probability ensemble. We find
$$S_m(g) \leq 2 \sum_{n=n_0}^{+\infty} \frac{f(g)}{\pi^2 n^2} \log(\frac{\pi^2 n^2}{f(g)}) + \log(2n_0 - 1).$$

- The sum is convergent, we choose an integer n_0 for optimal bound.

Strong coupling limit

- As we mentioned, sttring perturbation series is actually convergent. We can extrapolate to strong coupling $g \to \infty$.

- The optimal bound $n_0 \sim g^{2+\epsilon}$, we have

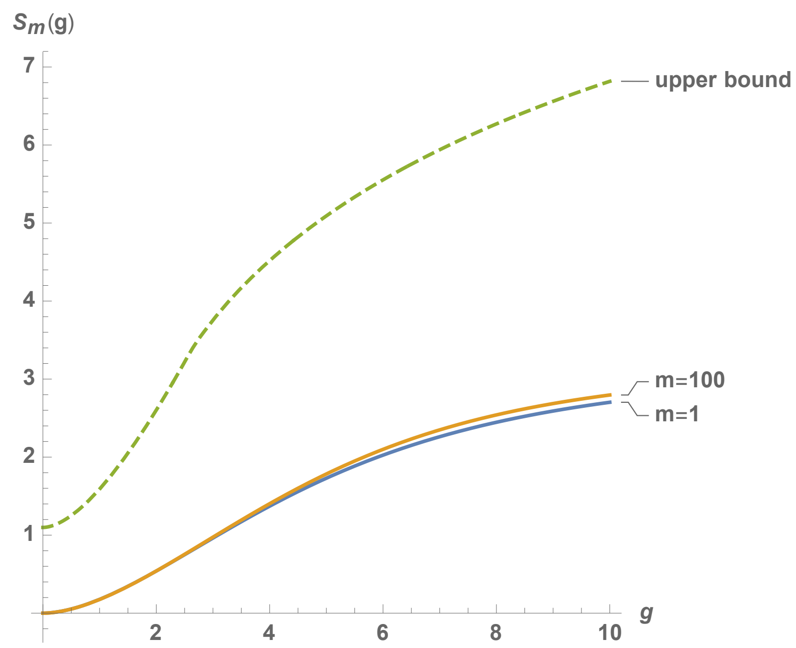
$$S_m(g) < (2+\epsilon)\log(g), \quad g \sim \infty. \tag{10}$$

- The logarithmic bound is universal, e.g. for BMN operator with 3 stringy modes $O^J_{(m_1,m_2,m_3)}$ with closed string level matching condition $m_1 + m_2 + m_3 = 0$, we have

$$S_{(m_1,m_2,m_3)}(g) < (6+\epsilon)\log(g), \quad g \sim \infty.$$

Some numerical analysis

- As an illustration example we plot the entropy $S_m(g)$ for for two case $m = 1, 100$ and $0 < g < 10$. We use the data up to genus 3 and also truncate the sum at $|n| < 10000$.



- $S_m(g)$ appears to be a monotonically increasing function of g, though it seems difficult to give a rigorous proof.
$S_m(g)$ seems to depends very weakly on the string mode m.

Comparing to Thermal Entropy

- The thermal mixed state in a quantum system can be obtained as the reduced state by tracing over one party in a thermofield double (TFD) state, which is a pure quantum state entangled between two similar systems. The von Neumann entropy of the thermal state is the entanglement entropy of the TFD state.

- Let $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ denote the eigenvalues of a Hamiltonian. For a thermal state, the probability at an excited state is $p_i = \frac{1}{Z} e^{-\beta\lambda_i}$, where $\beta \equiv \frac{1}{k_B T}$ is the inverse temperature and $Z = \sum_i e^{-\beta\lambda_i}$ is the partition function. The von Neumann entropy can be written as

$$S = -\sum_{i=1}^{\infty} p_i \log(p_i) = \log(Z) + \frac{\beta}{Z}\sum_i \lambda_i e^{-\beta\lambda_i} = \log(Z) + T\partial_T \log(Z). \tag{11}$$

- For example, as a warm-up exercise, for the simple case of a harmonic oscillator $\widehat{H} = \frac{\widehat{p}^2}{2} + \frac{\omega^2 \widehat{x}^2}{2}$, it is not difficult compute the von Neumann entropy of a thermal state exactly (set $\hbar = 1$)

$$S = \frac{\beta\omega}{e^{\beta\omega} - 1} - \log(1 - e^{-\beta\omega}). \tag{12}$$

- We will be interested in the high temperature limit where the probability is more evenly distributed between all excited states, so the entropy should be maximized. For the harmonic oscillator

$$S \sim \log(T), \qquad T \sim \infty. \tag{13}$$

- More generally, we will use the number of states defined by

$$N(\lambda) = \#\{j \in \mathbb{N} : \lambda_j < \lambda\}. \tag{14}$$

The partition function can be computed by

$$Z = \int_0^\infty e^{-\beta\lambda} dN(\lambda). \tag{15}$$

To compute the asymptotic behavior of the entropy, we only need the asymptotic behavior of $N(\lambda)$.

Calabi-Yau Quantum Mechanics

- We consider quantum systems derived from toric Calabi-Yau geometries, where the Hamiltonians are exponential functions of the position and momentum operators, e.g.

$$\begin{aligned} \widehat{H} &= e^{\widehat{x}} + e^{-\widehat{x}} + e^{\widehat{p}} + e^{-\widehat{p}}, \qquad && \mathbb{P}^1 \times \mathbb{P}^1 \text{ model}, \\ \widehat{H} &= e^{\widehat{x}} + e^{\widehat{p}} + e^{-\widehat{x}-\widehat{p}}, && \mathbb{P}^2 \text{ model}. \end{aligned} \tag{16}$$

- The exact quantization condition can be written in terms of refined topological string partition function, now known as the TS/ST (Topological String/Spectral Theory) correspondence.

- Some nice mathematical results on the asymptotics of the energy eigenvalues were proven in Laptev, Schimmer, and Takhtajan, 2016, 2019. The the number of states grow like

$$N(\lambda) \sim \log^2(\lambda), \qquad \lambda \sim \infty \tag{17}$$

where we have neglected the coefficient factor which depends on specific models.

- In the high temperature limit, the partition function is

$$Z \sim \int_1^\infty e^{-\beta\lambda} \frac{\log(\lambda)}{\lambda} d\lambda. \tag{18}$$

This integral can be represented by the Meijer G-function. Using its series expansion as well as some alternative elementary calculations, we find the leading asymptotic behavior

$$Z \sim \log^2(T), \qquad T \sim \infty. \tag{19}$$

- So as in the harmonic oscillator case, the first term in (11) dominates and we have

$$S \sim \log(Z) \sim \log(\log(T)), \qquad T \sim \infty. \tag{20}$$

So the entropy grow much slower than the harmonic oscillator in the high temperature limit.

No Finite Bound for Entropy

- We consider whether it is possible to have a finite upper bound for thermal entropy in the infinite temperature limit $T \to \infty$ for some quantum systems.

- We assume that the Hilbert space is infinite dimensional where the energy eigenvalues can be shifted to be all positive.
  Further assume the sum in the partition function is convergent for any finite temperature, so the partition function $Z(T)$ is well defined for any T, i.e. there is no exponential growth of the number of states $N(\lambda)$, unlike string theory.

- It is easy to check by taking derivative that both $Z(T)$ and $S(T)$ are monotonically increasing functions of T

$$S(T) = \log(Z) + T\partial_T \log(Z) > \log(Z). \tag{21}$$

  Since the Hilbert space is infinite dimensional, the partition function $Z(T)$ tends to infinity as $T \to \infty$, therefore the entropy also tends to infinity and there is no finite upper bound.

A Possible Cosmological Application

- A folklore of quantum gravity is the finiteness of entropy, in contrast to its divergence in generic calculations in quantum field theory.

- Of course, the dimension of Hilbert space is infinite in perturbative string theory but this is not necessarily in conflict with the folklore.
  In a countable infinite dimensional Hilbert space, the density matrix with finite entropy is a meager set in the usual topology.
  However, in reasonable physical situations, the entropy is usually still finite. A. Wehrl, General properties of entropy, Rev. Mod. Phys. 50, 221-260 (1978)

- Here we consider discrete spectrum (countable basis), because for continuous spectrum, the Hilbert space and von Neumann entropy are not mathematically well defined. (Instead it is called rigged Hilbert space.)

- For example, for a probability distribution that scales as power law $p_n \sim n^{-\alpha}$ among an orthogonal basis of states, the convergence of the sum $\sum_{n=1}^{\infty} p_n$ is equivalent to $\alpha > 1$, in which case the von Neumann entropy $-\sum_{n=1}^{\infty} p_n \log(p_n)$ is also finite.

- We may consider a more contrived probability distribution $p_n \sim \frac{1}{n \log^{\alpha}(n)}$. For $\alpha \leq 1$ the sum $\sum_{n=1}^{\infty} p_n$ is divergent, while for $\alpha > 2$ both sums $\sum_{n=1}^{\infty} p_n$ and $-\sum_{n=1}^{\infty} p_n \log(p_n)$ are convergent. So in this case in a limited range $1 < \alpha \leq 2$ we can have a probability distribution where the entropy is infinite.

- In the quantum models we studied, the entropy is indeed finite at a finite temperature, but tends to infinity in the infinite temperature limit.

- Is it possible to get a finite maximal entropy in a similar natural limit in some other situations?

- An important source of motivation comes from the finite cosmological event horizon of de Sitter space, which appears to be the current state of our universe. (horizon area $\sim$ entropy, Gibbons and Hawking, 1977.) Some discussions of dark energy.

- Consider a 4d de Sitter space with a cosmological constant $\Lambda > 0$. Some physical quantities (set $\hbar = c = 1$)
  The horizon size $\quad r \sim \Lambda^{-\frac{1}{2}}$,
  The vacuum energy density $\quad \rho_{vac} \sim m_p^2 \Lambda$,
  where $m_p \sim l_p^{-1}$ is the Planck mass.

- Astrophysical observations suggest

$$\frac{\rho_{vac}}{m_p^4} \sim \frac{\Lambda}{m_p^2} \sim \frac{l_p^2}{r^2} \sim 10^{-122} \tag{22}$$

- The famous cosmological constant problem "why it is so small" can be rephrased as:
  Why is the entropy of our universe so big $S \sim \frac{r^2}{l_p^2} \sim 10^{122}$?

Entropy of Our Universe

- In generic scenarios, the dominant contribution to entropy in observable universe comes from supermassive black holes (about 10^{93} each). There may be trillions of them, but the total entropy is still negligible comparing to the Gibbons-Hawking entropy of 10^{122}.
  For a recent update, see e.g. S. Profumo et al, A New Census of the Universe's Entropy, arXiv: 2412.11282.

- The entropy of our universe is finite ($\sim 10^{122}$) and probably will not increase much in the far future. Is this a hint for a universal finite upper bound for entropy in consistent theories of quantum gravity?

- Although the Hilbert space can be infinite dimensional, we are not aware of a robust example where the entropy could become arbitrarily large (even much larger than the entropy of our universe) in a reasonable physical process, taking into account the full effects of quantum gravity.

- The BMN strings provide such a test.

- For the BMN strings, it was found that at finite coupling g, the entropy is indeed also finite, while it is naively expected that as $g \to \infty$, the probability would be evenly distributed among the infinite dimensional Hilbert subspace, so the entropy should likely tend to infinity.

- Nevertheless, it would be a pleasant surprise if it turns out that the entropy of BMN strings does have a finite upper bound as $g \to \infty$. (Our bound (10) is not strong enough to decide.)
  Such a bound may be related to the entropy of our current universe, thus could provide a natural estimate of the cosmological constant.

- An encouraging hint is that for the Calabi-Yau models, which are related to topological string theory, a toy version of quantum gravity, the entropy does grow much slower (20) than the conventional models.

- To conclude, it would be interesting to settle this issue in the future.

Thank You