Quantum K-theory at roots of unity   w/ A. Smirnov   2412.19383

① p-curvature. Let p be prime, E - holomorphic vector bundle over $\mathbb{P}^1$

∇ - connection in E, $f \in \Gamma(E)$

$$C_p(\nabla) = \nabla^p f \mod p$$

Ex: $\nabla^2 f$, $\nabla = \partial_z + sA(z)$

$$(\partial + A)(\partial f + Af) = \partial^2 f + (\partial A) f + 2A\partial f + \cancel{A\partial f} + A^2 f$$

$$= \cancel{\partial^2 f} + \cancel{2A\partial f} + (\partial A + A^2) f \mod 2$$

Claim: $C_p(\nabla_i) \in \mathrm{Mat}_N(\mathbb{F}_p(z_1, \dots z_e)[s])$

Grothendieck – Katz (1971)

② Why p-curvature? Holonomic PDEs

Q: Given a full basis of solutions of ODE/PDE in $\mathbb{F}_p^n$ what are the conditions of existence of the full basis of analytic solutions?

Conj [GK]: Yes if $C_p(\nabla_i) = 0$ for all but a few p.

Ex: over $\mathbb{C}$: $y' = \frac{1}{az} y$, $a \in \mathbb{Z}$

$y = C z^{1/a}$ $\xleftarrow{?}$ over $\mathbb{F}_p$ Ansatz: $y = z^b$, $b \in \mathbb{Z}/p\mathbb{Z}$

$$b z^{b-1} = \frac{1}{a} z^{b-1} \mod p$$

$$ab \equiv 1 \pmod p$$

$p \nmid a$.

③ Main theorem [Etingof-Varchenko] [KSmirnov]

The following connections are isospectral

$$C_p(\nabla_i(z))[s] \quad \text{and} \quad (s^p - s)C(\nabla(z^p)) \pmod p$$

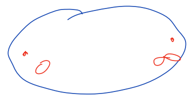
Frobenius $z \mapsto z^p$

periodic pencil of flat connections

$s \mapsto s+1$ $s^p - s$ is invariant mod p

$(a+b)^p \equiv a^p + b^p$.

④ Quantum K-theory (quasimap) Equivariant $\circlearrowleft T$

$\mathbb{C}_q^\times \circlearrowright$  $\xrightarrow{f^d} X \subset \mathcal{X}$ stack

$\mathbb{P}^1$

quiver variety
hyper Kähler

$\mathcal{F} \circledast \mathcal{G} = \mathcal{F} \otimes \mathcal{G} + \sum_d z^d \ldots$

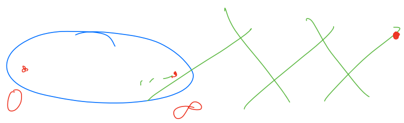
$T^*_\hbar \mathrm{Gr}_{k,n}$

$\mathbb{C}^k$, $\mathbb{C}^n$

$\mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n) \oplus \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^k) \oplus \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^k)$

$QM^d_{\text{nonsing}}(X)$ $\quad QM^d_{\text{relative}}(X)$ $\quad (\mathbb{C}^\times)^n_{a_1 \ldots a_n} \times \mathbb{C}^\times_\hbar$ $\quad \overline{GL(k;\mathbb{C})}$

$X = \mathbb{P}^1 \qquad z = \frac{u}{v} = \frac{(z-e_1)\cancel{(z-e_2)}\ldots(z-e_d)}{(z-f_1)\cancel{(z-f_2)}\ldots(z-f_d)}$



Capping operator: $\Psi(z,a,\hbar,q) \in K^{loc}_T(X)^{\otimes 2}[[z]]$

Vertex $\quad \hat{V}^{(\tau)}_{(z,a,\hbar)} = \Psi \cdot V^{(\tau)}(z,a,\hbar)$

$\Psi_{i,j} = V_j^{(S_i(z,a))}(z,a,q)$ 

$\Psi(z,a,\hbar)$ satisfies a q-difference equation (dynamical equation)

$$\Psi(zq^{\mathcal{L}}, a, q)\mathcal{L} = M_{\mathcal{L}}(z,a,q)\,\Psi(z,a,q), \quad \mathcal{L} \in \mathrm{Pic}(X) \cong \mathbb{Z}^\ell$$

limit $q \to 1$ $\qquad \mathcal{M}_{\mathcal{L}}(z,a) = M_{\mathcal{L}}(z,a,q)\big|_{q=1}$

Claim $\qquad [\mathcal{M}_{\mathcal{L}_1}, \mathcal{M}_{\mathcal{L}_2}] = 0 \qquad$ Lax matrix for XXZ chain

limit $q \to \zeta_p$
$\zeta_p^p = 1$
primitive root of 1

$$M_{\mathcal{L}^p}(z,a,q) = M(zq^{(p-1)\mathcal{L}}) \ldots M_{\mathcal{L}}(zq^{\mathcal{L}})\, M_{\mathcal{L}}(zq^{\mathcal{L}}) M_{\mathcal{L}}(z)$$

$$\mathcal{M}_{\mathcal{L},\zeta_p}(z,a) = M_{\mathcal{L}^p}(z,a,q)\big|_{q=\zeta_p}.$$

Theorem 1: Let $\{\lambda_i(z,a)\}$ be the set of eigenvalues of $\mathcal{M}_{\mathcal{L}}$ then $\{\lambda_i(z^p, a^p)\}$ is the set of eigenvalues of $\mathcal{M}_{\mathcal{L},\zeta_p}$.

Theorem 2: The operator $\Psi(z,a,q) \cdot \Psi(z^p, a^p, q^{p^2})^{-1}$

is well defined as $q^p \to 1$.

lesing q^{p^2}?

$\mathbb{C}[X,Y]$ $\quad XY = qYX$

$\mathbb{A}^2$

$\mathbb{F}_p$: $X \mapsto X^p$, $Y \mapsto Y^p$

$$X^p Y^p = q^{p^2} Y^p X^p$$

Ex: $X = T^* \mathbb{P}^0$ $\quad \Psi(qz) = \frac{1-z}{1-\hbar z} \Psi(z)$

$$\Psi(z) = \exp\left(\sum_{m=1}^{\infty} \frac{1-\hbar^m}{m(1-q^m)} z^m \right)$$

Theorem 3: Let $F(z,a,\zeta_p) = \Psi(z,a,q) \cdot \Psi(z^p, a^p, q^{p^2})^{-1} \Big|_{q^p=1}$

Then

$$\underbrace{\mathcal{M}_{\mathcal{L},\zeta_p}(z,a)} = F(z,a,\zeta_p) \cdot \underbrace{\mathcal{M}_{\mathcal{L}}(z^p,a^p)} \cdot F(z,a,\zeta_p)^{-1}$$

isospectral.

⑤ Number theory limit

a) Cohomology limit $\quad \frac{\Psi(qz_i)}{\Psi(z_i)} \xrightarrow[\epsilon \to 0]{} z_i \partial_{z_i} - \zeta C_i(z,a)$

$\log q / \log \hbar$

chern classes

$$q = e^{\epsilon} = 1 + \epsilon + O(\epsilon^2)$$

$$q^{z_i \partial_{z_i}} = 1 + \epsilon z_i \frac{\partial}{\partial z_i} + \dots$$

b) reduce to $\mathbb{Q}_p$ - p-adic numbers.

$$|p|_p = \frac{1}{p}$$

ζ_p $\quad \not\in$ $\quad a_0 + a_1 p + a_2 p^2 + \dots$

$\mathbb{Z}/\mathbb{Z}p$

$$p = 7$$

$$1 + 7 + 7^2 + 7^3 + \dots \text{ "=" } \frac{1}{1-7} = -\frac{1}{6}$$

$\mathbb{Q}_p[\pi]$, $\quad \pi^{p-1} = -p$, $\quad |\pi|_p = \frac{1}{p^{\frac{1}{p-1}}} < 1$

$$\zeta_p = 1 + a\pi + \pi^2 + \dots$$

$\mathbb{Z}/p\mathbb{Z} \subset \mathbb{Q}_p$ $\quad \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} \Big/ (\pi)$ - field.

maximal

LHS $M_{\mathcal{L},\zeta_p}(z,a) = \left(M_{\mathcal{L}}(z)\, q^{z\frac{\partial}{\partial z}} \right)^p = 1 + \pi^p(\nabla_1^p - \nabla_1) + O(\pi^{p+1})$

$$\frac{M_{\mathcal{L},\zeta_p} - 1}{\pi^p} \equiv C_p(\nabla) \pmod{\pi} \pmod{p}$$

RHS

$$M_{\mathcal{L}}(z^p) = 1 + C(\nabla(z^p))\,\pi^p(s^p - s) \pmod{p}$$

$$\frac{M_{\mathcal{L}}(z^p) - 1}{\pi^p} \equiv (s^p - s)\,C.(z^p) \mod p.$$

From Theorem 3 main Theorem follows.