The space of gauge invariant operators at finite N

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Motivation: Gravity Hilbert Space from CFT

- Single traces operators in CFT generated a *bosonic Fock space* at leading order, with Fock oscillators identified with single trace operators

$$a^\dagger_\alpha \longleftrightarrow \mathcal{O}_\alpha, \qquad a^\dagger_{\alpha_1}\cdots a^\dagger_{\alpha_k}|0\rangle \longleftrightarrow \mathcal{O}_{\alpha_1}\cdots\mathcal{O}_{\alpha_k}|0\rangle,$$

  At large N there is no cut off on ocupation numbers of Fock space states.
- Interactions among these "particles" are $O(1/N)$ effects, mapping to bulk gravitational couplings $G_N \sim 1/N^2$.
- At finite N we can have *giant graviton branes* represented by complex multi-trace operators known as Schur polynomials. These operators have a dimension of order N.
- At finite N we can have Lin-Lunin-Maldacena geometries represented by operators with a dimension of order N^2.

The space of gauge invariant operators in the gauge theory reproduces the Hilbert space of the holographic gravitational theory. This talk will describe the structure of this space of gauge invariant operators.

Motivation: The Ring of Matrix Invariants

Consider the quantum mechanics of a single $N \times N$ matrix, with a gauge symmetry

$$X \to U^\dagger X U \qquad\qquad U \in U(N)$$

At $N = \infty$ the space of gauge invariant operators is **freely** generated by

$$\phi_k \quad = \quad \mathrm{Tr}(X^k)$$

Complete space is arbitrary polynomials in the ϕ_k and generators ϕ_k are algebraically independent.

At finite N there are **trace relations**. *Any* 2×2 matrix X obeys

$$2\mathrm{Tr}(X^3) - 3\mathrm{Tr}(X^2)\mathrm{Tr}(X) + \mathrm{Tr}(X)^3 \quad = \quad 0 \;=\; 2\phi_3 - 3\phi_2\phi_1 + \phi_1^3$$

so the generators are no longer free - they obey non-trivial relations.

How does this modify the space of gauge invariant operators?

Gauge invariant operators are freely generated by N invariants

$$\phi_k(X) = \mathrm{Tr}(X^k) \qquad k = 1, 2, \cdots, N$$

Basic Question: The Ring of Matrix Invariants

Consider quantum mechanics of d matrices X^a, $a = 1, 2, \cdots, d$, with gauge symmetry

$$X^a \to U^\dagger X^a U$$

At $N = \infty$ the space of gauge invariant operators is freely generated by

$$\phi_{\{a_1,a_2,\ldots a_k\}} \quad = \quad \mathrm{Tr}(X^{a_1} X^{a_2} \cdots X^{a_k})$$

The complete space is given by arbitrary polynomials in the $\phi_{\{a_1,a_2,\ldots a_k\}}$ and the generators $\phi_{\{a_1,a_2,\ldots a_k\}}$ are algebraically independent.

How is this description modified at finite N?

The goal of this talk is to answer this question by solving the complete set of trace relations for multimatrix models.

Outline

1. Motivational Counting Problem
2. Statement of the Result
3. Problem Statement
4. Warm up: one matrix
5. Molien Weyl Formula
6. Two and Three Matrix Model
7. General Lessons

Counting polynomials

How many degree d polynomials can be constructed using x and y?

degree 0	1	1
degree 1	2	$x,\ y$
degree 2	3	$x^2,\ xy,\ y^2$
degree 3	4	$x^3,\ x^2y,\ xy^2,\ y^3$
$\vdots$	$\vdots$	$\vdots$

We can generate all of these polynomials, exactly once, as follows:

$$\begin{aligned} \frac{1}{1-x}\frac{1}{1-y} &= (1+x+x^2+\cdots)(1+y+y^2+\cdots) \\ &= 1 \quad +x+y \quad +x^2+xy+y^2 \quad +\cdots \end{aligned}$$

To count set $x = t = y$ to obtain

$$\left(\frac{1}{1-t}\right)^2 = 1+2t+3t^2+4t^3+\cdots$$

The function

$$H(t) = \frac{1}{(1-t)^2} = \sum_{n=0}^{\infty} c_n t^n$$

Is called a *Hilbert series*. The integer c_n counts how many degree n polynomials can be freely generated from the 2 variables x and y.

Some obvious generalizations

Generic number of generators and degrees: How many degree d polynomials can be freely generated using x_1, x_2, x_3^2 and x_4^3?

$$H(t) = \frac{1}{(1-t)^2(1-t^2)(1-t^3)} = \sum_{n=0}^{\infty} c_n t^n$$

Counting with constraints: Spherical harmonics in $\mathbb{R}^d$: count degree n polynomials constructed using $x^1, x^2, \cdots, x^d$ subject to the constraint

$$(x^1)^2 + (x^2)^2 + \cdots + (x^d)^2 = 1$$

$$H(t) = \frac{1-t^2}{(1-t)^d} = \sum_{n=0}^{\infty} c_n t^n$$

In general the numerator can be a polynomial with positive and negative signs, reflecting constraints and relations between constraints.

Last Counting Problem

How many degree d polynomials can be constructed using $\{x, y\}$ freely and using exactly one of $\{1, w^2, v^3\}$?

We can generate all of these polynomials, exactly once, as follows:

$$1 \times \frac{1}{1-x}\frac{1}{1-y} \;+\; w^2 \times \frac{1}{1-x}\frac{1}{1-y} \;+\; v^3 \times \frac{1}{1-x}\frac{1}{1-y}$$

Thus to count we can again set $x = y = w = v = t$ to obtain

$$\frac{1+t^2+t^3}{(1-t)^2} \;=\; 1 + 2t + 4t^2 + 7t^3 + \cdots$$

matching x, y degree 1, x^2, xy, y^2, w^2 degree 2 and $x^3, x^2y, xy^2, y^3, w^2x, w^2y, v^3$ degree 3.

The Hilbert series is now

$$H(t) = \frac{1+t^2+t^3}{(1-t)^2} = \sum_{n=0}^{\infty} c_n t^n$$

c_n again counts how many degree n polynomials can be freely generated from the 2 variables x and y, and using exactly one of $\{1, w^2, v^3\}$.

Key Message of Last Counting Problem

How many degree d polynomials can be constructed using $\{x, y\}$ freely and using exactly one of $\{1, w^2, v^3\}$?

The Hilbert series is

$$H(t) = \frac{1 + t^2 + t^3}{(1-t)^2} = \sum_{n=0}^{\infty} c_n t^n$$

c_n again counts how many degree n polynomials can be freely generated from the 2 variables x and y, and using w^2 and v^3 each at most once.

The denominator tells us about generators that act freely.

The numerator tells us about generators that act at most once.

$$1 \times x^n y^m \;\cup\; w^2 \times x^n y^m \;\cup\; v^3 \times x^n y^m$$

Why count ? (from Partition Functions to Hilbert Series)

We compute partition functions of multi-matrix harmonic oscillators

$$Z(x) = \sum_{\text{states } i} e^{-\beta E_i}$$

Solve eigenproblem as usual: use matrix creation operators

$$A^{a\dagger} = X^a - i\Pi^a$$

Energy eigenstates are (*any* trace structure)

$$|E\rangle = \mathrm{Tr}(A^{a_1\dagger} A^{a_2\dagger} \cdots A^{a_n\dagger})|0\rangle$$

Energy $E = n =$ degree of polynomial (after subtracting off groundstate energy)

Partition function = sum over states = a sum over polynomials

Graded by $e^{-\beta E} = x^n$ $(x = e^{-\beta})$ = degree of the polynomial.

The partition function is a Hilbert series!

Our Basic Result

Partition functions of multi-matrix model harmonic oscillators for matrices X^a, $a = 1, 2, \cdots, d$ take the form $(x = e^{-\beta})$

$$Z(x) = \sum_{\text{states } i} e^{-\beta E_i} = \frac{1 + \sum_i c_i^s x^i}{\prod_j (1 - x^j)^{c_j^p}}.$$

Each factor in the denominator is associated to an operator of degree j

$$x^j \leftrightarrow P_A = \mathrm{Tr}(X^{a_1} X^{a_2} \cdots X^{a_j}) + \cdots$$

and is called a *primary invariant*. A takes $N_P = \sum_j c_j^p$ values.

Each monomial x^i in the numerator is associated with an operator of degree i

$$x^i \leftrightarrow S_B = \mathrm{Tr}(X^{a_1} X^{a_2} \cdots X^{a_k}) \mathrm{Tr}(X^{a_{k+1}} \cdots X^{a_i}) + \cdots$$

and is called a *secondary invariant*. B takes $N_S = 1 + \sum_i c_i^s$ values, where we set $S_1 = 1$.

The space of all gauge invariant operators (loop space) takes the form

$$\mathcal{H} \quad = \quad \bigoplus_{B=1}^{N_S} \prod_{A=1}^{N_P} \sum_{\{n_A\}=0}^{\infty} (P_A)^{n_A} S_B$$

Basic Result continued

The complete space of gauge invariant operators (loop space) is a free module generated by the primary and secondary invariants

$$\mathcal{H} \;=\; \bigoplus_{B=1}^{N_S} \prod_{A=1}^{N_P} \sum_{\{n_A\}=0}^{\infty} (P_A)^{n_A} S_B$$

This has a natural physical interpretation

$$\mathcal{H} \;=\; (P_1)^{n_1}(P_2)^{n_2}\cdots(P_{N_P})^{n_{N_P}} S_1 \bigoplus (P_1)^{n_1}(P_2)^{n_2}\cdots(P_{N_P})^{n_{N_P}} S_2$$
$$\bigoplus\cdots\bigoplus (P_1)^{n_1}(P_2)^{n_2}\cdots(P_{N_P})^{n_{N_P}} S_{N_S}$$

as a Fock space of perturbative degrees of freedom constructed on the non-perturbative states/backgrounds of the theory.

In the ring of gauge invariants, primary invariants correspond to perturbative degrees of freedom while the secondary invariants correspond to non-perturbative states of the theory.

Matrix Quantum Mechanics; Loop Space

We study matrix quantum mechanics of d $N \times N$ Hermittian matrices

$$H = \frac{1}{2}\sum_{a=1}^{d} \mathrm{Tr}(\Pi^a \Pi^a) + \frac{1}{2}\sum_{a=1}^{d} \mathrm{Tr}(X^a X^a)$$

$$[\Pi^a_{ij}(t), X^b_{kl}(t)] = -i\delta_{il}\delta_{jk}\delta^{ab}$$

The model has a $U(N)$ gauge symmetry

$$X^a \to U^\dagger X^a U \qquad \Pi^a \to U^\dagger \Pi^a U$$

Complete space of gauge invariant operators is given by traces of words constructed from the X^a's as follows

$$\mathrm{Tr}(X^1 X^2 X^2 X^1 X^3 X^6 \cdots)$$

What is a complete description of this *loop space*?

Warm up: one matrix

We can diagonalize the Hamiltonian by introducing the creation and annihilation operators

$$A^\dagger \quad = \quad X - i\Pi$$

Up to a ground state energy

$$H \quad = \quad \mathrm{Tr}(A^\dagger A)$$

For a single matrix the partition function is ($x = e^{-\beta}$, $E_n = n$)

$$Z(x) \quad = \quad \frac{1}{(1-x)(1-x^2)(1-x^3)\cdots(1-x^N)}$$

This reproduces our earlier claim: the complete space of gauge invariant operators is freely generated by the N invariants

$$\phi_k(X) = \mathrm{Tr}(X^k) \qquad k = 1, 2, \cdots, N$$

Understanding trace relations

Trace relations for 2×2 matrices have $(2+1)!$ terms: sum terms that result by **anti-symmetrizing** column indices: (Procesi, 1975)

$$(W_1)_{i_1 i_1}(W_2)_{i_2 i_2}(W_3)_{i_3 i_3} - (W_1)_{i_1 i_2}(W_2)_{i_2 i_1}(W_3)_{i_3 i_3} - (W_1)_{i_1 i_1}(W_2)_{i_2 i_3}(W_3)_{i_3 i_2}$$

$$-(W_1)_{i_1 i_3}(W_2)_{i_2 i_2}(W_3)_{i_3 i_1} + (W_1)_{i_1 i_2}(W_2)_{i_2 i_3}(W_3)_{i_3 i_1} + (W_1)_{i_1 i_3}(W_2)_{i_2 i_1}(W_3)_{i_3 i_2} = 0$$

W_1, W_2, W_3 are 2×2 matrices. Antisymmetrizing $2+1=3$ indices that take 2 values gives zero.

W's constructed using a single matrix give trace relations for a single matrix model.

W's constructed using d matrices give trace relations for matrix model with d species of matrix.

Trace relations are equally applicable for bosonic and fermionic matrices, for any coupling strengths and any interactions. Only important parameter is N.

Understanding trace relations

Trace relations for $N \times N$ matrices have $(N+1)!$ terms: sum terms that result by **anti-symmetrizing** column indices: (Procesi, 1975)

$$(W_1)_{i_1 i_1}(W_2)_{i_2 i_2}(W_3)_{i_3 i_3}\cdots(W_{N+1})_{i_{N+1} i_{N+1}} - (W_1)_{i_1 i_2}(W_2)_{i_2 i_1}(W_3)_{i_3 i_3}\cdots(W_{N+1})_{i_{N+1} i_{N+1}}$$

$$-\cdots+(W_1)_{i_1 i_2}(W_2)_{i_2 i_3}(W_3)_{i_3 i_1}\cdots(W_{N+1})_{i_{N+1} i_{N+1}}+\cdots=0$$

$W_1, W_2, \cdots W_{N+1}$ are $N \times N$ matrices. Antisymmetrizing $N+1$ indices that take N values gives zero.

W's constructed using a single matrix give trace relations for a single matrix model.

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Trace relations are equally applicable for bosonic and fermionic matrices, for any coupling strengths and any interactions. Only important parameter is N.

Motivation: The Ring of Matrix Invariants

For $N = 2$ we have

$$\mathrm{Tr}(W_1)\mathrm{Tr}(W_2)\mathrm{Tr}(W_3) - \mathrm{Tr}(W_1W_2)\mathrm{Tr}(W_3) - \mathrm{Tr}(W_1W_3)\mathrm{Tr}(W_2) - \mathrm{Tr}(W_2W_3)\mathrm{Tr}(W_1)$$

$$+\mathrm{Tr}(W_1W_2W_3) + \mathrm{Tr}(W_1W_3W_2) = 0$$

For $W_1 = X^k$, $W_2 = X = W_3$ we get: $\phi_k\phi_1^2 - 2\phi_{k+1}\phi_1 - \phi_2\phi_k + 2\phi_{k+2} = 0$

$\Rightarrow$ we obtain ϕ_k with $k \geq 3$ as polynomials in ϕ_1, ϕ_2.

$$\begin{aligned} \phi_3 &= \frac{3}{2}\phi_2\phi_1 - \frac{1}{2}\phi_1^3 \\ \phi_4 &= \phi_3\phi_1 + \frac{1}{2}\phi_2^2 - \frac{1}{2}\phi_1^2\phi_2 \\ &\vdots \end{aligned}$$

At finite N the space of gauge invariant operators is freely generated by $\phi_k = \mathrm{Tr}(X^k)$ $k = 1, 2, \cdots, N$.

Molien-Weyl Formula

Count gauge invariant operators constructed from adjoint bosonic fields with energies E_i:

$$Z(\beta)=\sum_{n_1=0}^{\infty}x^{n_1E_1}\sum_{n_2=0}^{\infty}x^{n_2E_2}\cdots\times\#(n_1,n_2,...)$$

where $x=e^{-1/T}=e^{-\beta}$ and $\#(n_1,n_2,...)$ is the number of singlets in the tensor product $\mathrm{sym}^{\mathrm{n_1}}_{\mathrm{adj}}\otimes\mathrm{sym}^{\mathrm{n_2}}_{\mathrm{adj}}\otimes\cdots$.

$$\begin{aligned}\#(n_1,n_2,...) &= \int_{U(N)}[DU]\ \chi_{\mathrm{sym}^{\mathrm{n_1}}_{\mathrm{adj}}\otimes\mathrm{sym}^{\mathrm{n_2}}_{\mathrm{adj}}\otimes\cdots}(U)\chi_{\mathrm{singlet}}(U)\\ Z(\beta) &= \int_{U(N)}[DU]\ \prod_i\sum_{n_i=0}^{\infty}x^{n_iE_i}\chi_{\mathrm{sym}^{\mathrm{n_i}}_{\mathrm{adj}}}(U)\end{aligned}$$

Reduce the matrix integral above to an integral over the eigenvalues of U

$$Z(x)=\frac{(Z_{N=1}(x))^N}{(2\pi i)^{N-1}}\oint_{|t_1|=1}\frac{dt_1}{t_1}\cdots\oint_{|t_{N-1}|=1}\frac{dt_{N-1}}{t_{N-1}}\prod_{1\le k\le r\le N-1}\frac{1-t_{k,r}}{f_{k,r}}$$

where $t_{k,r}=t_kt_{k+1}\cdots t_r$ and

$$Z_{N=1}(x)\ =\ \frac{1}{\prod_i(1-x^{E_i})}\qquad f_{k,r}\ =\ \prod_{i=1}^{d}(1-x^{E_i}t_{k,r})(1-x^{E_i}t_{k,r}^{-1})$$

Two matrix model, $N = 2$

Graded partition function: (call X^1 as X and X^2 as Y and introduce a chemical potential for each: $x = e^{-\beta-\mu_X}$, $y = e^{-\beta-\mu_Y}$)

$$Z(x,y) = \frac{1}{(1-x)(1-y)(1-x^2)(1-xy)(1-y^2)}.$$

Generators:

$$m_1 = \mathrm{Tr}(X), \qquad m_2 = \mathrm{Tr}(Y),$$
$$m_3 = \mathrm{Tr}(X^2), \quad m_4 = \mathrm{Tr}(XY), \quad m_5 = \mathrm{Tr}(Y^2),$$

All trace relations from "Cayley-Hamilton". For $N = 2$: $T_2(A,B,C) = 0$ where

$$\begin{aligned} T_2(A,B,C) &= \mathrm{Tr}(A)\mathrm{Tr}(B)\mathrm{Tr}(C) - \mathrm{Tr}(AB)\mathrm{Tr}(C) - \mathrm{Tr}(AC)\mathrm{Tr}(B) \\ &\quad -\mathrm{Tr}(A)\mathrm{Tr}(BC) + \mathrm{Tr}(ABC) + \mathrm{Tr}(ACB), \end{aligned}$$

and A, B and C are any words constructed using X, Y as letters.

To give a trace relation, give A, B and C.

Words of length 3

Consider single trace operators constructed using n X matrices and m Y matrices.

For $m+n=3$, each choice of (m,n) gives a single trace relation and there is only one gauge-invariant single trace operator. All operators are produced by our generating set.

Example: Choose $m=2$ and $n=1$. We obtain

$$T_2(X,Y,Y)=\mathrm{Tr}(X)\mathrm{Tr}(Y)^2-2\mathrm{Tr}(XY)\mathrm{Tr}(Y)-\mathrm{Tr}(X)\mathrm{Tr}(XY)+2\mathrm{Tr}(XY^2)=0,$$

which implies

$$\mathrm{Tr}(XY^2)=\frac{1}{2}\Big(2m_2m_4+m_1m_5-m_1m_2^2\Big).$$

Swapping X and Y gives

$$\mathrm{Tr}(YX^2)=\frac{1}{2}\Big(2m_1m_4+m_2m_3-m_1^2m_2\Big).$$

We know

$$\mathrm{Tr}(X^3)=\frac{3}{2}\mathrm{Tr}(X^2)\mathrm{Tr}(X)-\frac{1}{2}\mathrm{Tr}(X)^3=\frac{3}{2}m_3m_1-\frac{1}{2}m_1^3 \qquad \mathrm{Tr}(Y^3)=\frac{3}{2}m_5m_1-\frac{1}{2}m_2^3$$

Words of length 4

As $m+n$ increases, number of distinct gauge-invariant operators increases.

Consider $m=2=n$: two independent operators, $\mathrm{Tr}(X^2Y^2)$ and $\mathrm{Tr}(XYXY)$, can be constructed.

Fortunately there are two independent trace relations. $T_2(Y^2,X,X)=0$ gives

$$\mathrm{Tr}(X^2Y^2)=\frac{1}{2}\Big(m_3m_5+2m_1m_2m_4+m_1^2m_5-m_1^2m_2^2-m_1^2m_5\Big).$$

$T_2(XY,X,Y)=0$ implies

$$\mathrm{Tr}(XYXY)=\frac{1}{2}\Big(m_2^2m_3+2m_4^2-m_3m_5+m_1^2m_5-m_1^2m_2^2\Big).$$

Crucially, the growth in the number of independent operators is matched by the emergence of additional trace relations.

Is this true for any m,n?

All Operators

We will prove that all gauge invariant operators can be written in terms of our generating set. Proof proceeds by induction.

Induction Hypothesis: Single-trace loops containing at most k matrices are determined by our generating set using the trace relations. (established for $k \leq 4$)

Using this hypothesis, prove all single trace loops containing $k+1$ matrices in the trace are determined by our generating set using the trace relations.

Consider the loop $\mathrm{Tr}(X^{n_1}Y^{m_1})$ with $n_1 + m_1 = k + 1$ for $k \geq 4$. At least one of n_1 or m_1 must be greater than 1. Without loss of generality, assume $n_1 > 1$. The trace relation for $A = X$, $B = X^{n_1-1}$, and $C = Y^{m_1}$ is

$$2\mathrm{Tr}(X^{n_1}Y^{m_1}) - \mathrm{Tr}(X)\mathrm{Tr}(X^{n_1-1}Y^{m_1}) - \mathrm{Tr}(X^{n_1})\mathrm{Tr}(Y^{m_1})$$

$$-\mathrm{Tr}(XY^{m_1})\mathrm{Tr}(X^{n_1-1}) + \mathrm{Tr}(X)\mathrm{Tr}(X^{n_1-1})\mathrm{Tr}(Y^{m_1}) = 0.$$

By the induction hypothesis, every term in this equation except the first contains at most k matrices in the trace and is thus expressible in terms of our generating set.

This establishes that $\mathrm{Tr}(X^{n_1}Y^{m_1})$ can also be expressed in terms of these variables. The same argument applies, with trivial changes, in the case where $m_1 > 1$.

All Operators

Consider $\mathrm{Tr}(X^{n_1}Y^{m_1}X^{n_2}Y^{m_2}\cdots X^{n_q}Y^{m_q})$, with

$$n_1+m_1+\cdots+n_q+m_q=k+1.$$

Call invariants, with q alternating $X^{\#}Y^{\#}$ blocks, type-q invariants.

Trace relation obtained from $A=X^{n_1}$, $B=Y^{m_1}$, and $C=X^{n_2}Y^{m_2}\cdots X^{n_q}Y^{m_q}$, is

$$\mathrm{Tr}(X^{n_1}Y^{m_1}\cdots X^{n_q}Y^{m_q})+\mathrm{Tr}(X^{n_1+n_2}Y^{m_2}\cdots X^{n_q}Y^{m_q+m_1})$$

$$-\mathrm{Tr}(X^{n_1})\mathrm{Tr}(X^{n_2}Y^{m_2}\cdots X^{n_q}Y^{m_q+m_1})-\mathrm{Tr}(Y^{m_1})\mathrm{Tr}(X^{n_1+n_2}Y^{m_2}\cdots X^{n_q}Y^{m_q})$$

$$-\mathrm{Tr}(X^{n_1}Y^{m_1})\mathrm{Tr}(X^{n_2}Y^{m_2}\cdots X^{n_q}Y^{m_q})+\mathrm{Tr}(X^{n_1})\mathrm{Tr}(Y^{m_1})\mathrm{Tr}(X^{n_2}Y^{m_2}\cdots X^{n_q}Y^{m_q})=0.$$

Second and third lines contain at most k matrices in a trace. By the induction hypothesis they are expressible in terms of our generating set.

First term is type-q invariant. Second term is type-$(q-1)$ invariant. We established type-1 invariant $\mathrm{Tr}(X^{n_1}Y^{m_1})$ can be expressed in terms of our generating set $\Rightarrow$ the type-2 invariant $\mathrm{Tr}(X^{n_1}Y^{m_1}X^{n_2}Y^{m_2})$ can also be expressed in terms of these variables.

This reasoning extends recursively, proving all type-q invariants are determined in terms of our generators.

Beyond the oscillator

Two important points to be aware of:

1. Our argument that the Hilbert series computes a partition function holds only for the oscillator. It uses the spectrum of the oscillator. It gives a guess for the generating invariants.

2. We proved the generating invariants span the space of gauge invariant operators using only trace relations. That these generators generate loop space freely is true for any interaction and any coupling strengths!

Three matrix model, $N = 2$

$$Z(x,y,z) = \frac{1 + xyz}{(1-x)(1-y)(1-z)(1-x^2)(1-y^2)(1-z^2)(1-xy)(1-xz)(1-yz)}.$$

Terms in the denominator correspond to the set of *primary* invariants

$$\begin{aligned} m_1 &= \mathrm{Tr}(X), \qquad m_2 = \mathrm{Tr}(Y), \qquad m_3 = \mathrm{Tr}(Z), \\ m_4 &= \mathrm{Tr}(X^2), \qquad m_5 = \mathrm{Tr}(Y^2), \qquad m_6 = \mathrm{Tr}(Z^2), \\ m_7 &= \mathrm{Tr}(XY), \qquad m_8 = \mathrm{Tr}(YZ), \qquad m_9 = \mathrm{Tr}(ZX), \end{aligned}$$

The terms in the numerator correspond to the *secondary* invariants

$$\{1, s\} = \{1, \mathrm{Tr}(XYZ)\}$$

Primary invariants act freely - they can be raised to any power. The secondary invariants are *quadratically reducible* and appear at most linearly, if at all.

Proof that s is quadratically reducible

When constructing the complete space of gauge invariant observables, we should not allow s to act more than linearly - **to avoid redundancy!**

Any action of s^2 can be replaced by an action of the primary invariants and terms with s appearing at most linearly, thanks to the constraint

$$
\begin{aligned}
& s^2 + s\,(m_1 m_2 m_3 - m_1 m_8 - m_2 m_9 - m_3 m_7) + 1\left[\frac{1}{4} m_1^2 m_2^2 m_3^2 - \frac{1}{2} m_1^2 m_2 m_3 m_8\right. \\
& -\frac{1}{4} m_1^2 m_5 m_6 + \frac{m_1^2 m_8^2}{2} - \frac{1}{2} m_1 m_2^2 m_3 m_9 - \frac{1}{2} m_1 m_2 m_3^2 m_7 + \frac{1}{2} m_1 m_2 m_6 m_7 \\
& +\frac{1}{2} m_1 m_3 m_5 m_9 - \frac{1}{4} m_2^2 m_4 m_6 + \frac{m_2^2 m_9^2}{2} + \frac{1}{2} m_2 m_3 m_4 m_8 - \frac{1}{4} m_3^2 m_4 m_5 + \frac{m_3^2 m_7^2}{2} \\
& \left. + \frac{m_4 m_5 m_6}{2} - \frac{m_4 m_8^2}{2} - \frac{m_5 m_9^2}{2} - \frac{m_6 m_7^2}{2} + m_7 m_8 m_9 \right] = 0.
\end{aligned}
$$

Complete space of gauge invariant operators

Complete space of gauge invariant operators is thus given by the direct sum of the space

$$m_1^{n_1} m_2^{n_2} m_3^{n_3} m_4^{n_4} m_5^{n_5} m_6^{n_6} m_7^{n_7} m_8^{n_8} m_9^{n_9} \times 1$$

and the space

$$m_1^{n_1} m_2^{n_2} m_3^{n_3} m_4^{n_4} m_5^{n_5} m_6^{n_6} m_7^{n_7} m_8^{n_8} m_9^{n_9} \times s$$

It is natural to interpret the first space above as perturbative excitations of the vacuum state, created by the identity.

It is natural to interpret the second space above as perturbative excitations of the non-trivial state, created by the secondary invariant s.

Thus the primary invariants play the role of perturbative degrees of freedom. Acting with the primary invariants is creating perturbative excitations. The secondary invariants play the role of non-trivial states (like a soliton). The state created by the secondary invariant can support perturbative excitations.

Hironaka Decomposition

All partition functions we compute take the form

$$Z(x) = \frac{1 + \sum_i c_i^s x^i}{\prod_j (1 - x^j)^{c_j^m}}.$$

This is the Hilbert series of the invariant ring $C_{N,d}$ of $GL(N)$ invariants of d $N \times N$ matrices. It matches the structure of the Hironaka decomposition.

The denominator encodes *primary* invariants, while the numerator encodes *secondary* invariants.

The number of primary invariants equals the number of denominator factors and gives the Krull dimension of the ring: $(d-1)N^2 + 1$.

That our partition functions all take the Hironaka form is key to our analysis. The Hochster-Roberts theorem ensures that $C_{N,d}$ is Cohen-Macaulay, since $GL(N)$ is a linearly reductive group over a field of characteristic zero. Thus, the ring admits a Hironaka decomposition, i.e., it is a free module over a polynomial subalgebra.

Counting Invariants: Experimental Results

For a matrix model with two matrices, the number of primary and secondary invariants counted as a function of N are given below.

N	Primary Invariants	Secondary Invariants
2	5	1
3	10	2
4	17	64
5	26	15,424
6	37	312,606,720
7	50	21,739,438,196,736

Table: Growth in the number of invariants as N increases.

Growth in the number of primary invariants is $N^2 + 1$.

Growth in secondary invariants is much more rapid than a power.

Counting Invariants

Alternative basis for the oscillator eigenstates: the restricted Schur polynomials $\chi_{R,(r,s)\alpha\beta}(A_1^\dagger, A_2^\dagger)$.

For a restricted Schur constructed with n $A_1^\dagger$'s and m $A_2^\dagger$'s we know that

1. The state has energy $E = n + m$.
2. **Young diagram labels:** $R \vdash n + m$, $r \vdash n$ and $s \vdash m$.
3. **Multiplicity labels:** $\alpha, \beta = 1, 2, \cdots, f_{rsR}$

Partition function

$$Z(x) = \sum_{R,r,s} (f_{rsR})^2 x^{n+m}$$

is lower bounded with asymptotics of Littlewood-Richardson number: For $n, m, n + m \sim \alpha N^2$ largest Littlewood-Richardson is (Pak, Panova, Yeliussizov, (2019))

$$f_{Rrs} = e^{\frac{\alpha N^2}{2} \log 2 + O(N)}$$

$$\Rightarrow [x^{\alpha N^2}] Z(x) > e^{\alpha N^2 \log 2}$$

Counting Invariants

Counting restricted Schurs, we lower bounded $Z(x)$

$$[x^{\alpha N^2}]Z(x) = [x^{\alpha N^2}]\frac{1+\sum_i c_i^s x^i}{\prod_j (1-x^j)^{c_j^p}} > e^{\alpha N^2 \log 2}$$

We can upper bound the coefficient $d_{\alpha N^2}$:

$$Z_p(x) \quad = \quad \frac{1}{\prod_j (1-x^j)^{c_j^p}} \quad = \quad \cdots + d_{\alpha N^2}\, x^{\alpha N^2} + \cdots$$

Introduce (note $C_k = \sum_{i=1}^{k-1} c_i^p$ and $\hat{d}_{\alpha N^2} > d_{\alpha N^2}$)

$$\hat{Z}_P(x) \quad = \quad \frac{1}{(1-x)^{C_k}}\frac{1}{(1-x^k)^{N^2+1-C_k}} \quad = \quad \cdots \hat{d}_{\alpha N^2}\, x^{\alpha N^2} + \cdots$$

Simple evaluation provides a value for $\hat{d}_{\alpha N^2} > d_{\alpha N^2}$.

$$[x^{\alpha N^2}]\hat{Z}_P(x) \quad = \quad e^{\frac{1}{k}((\alpha+k)\log(1+\frac{\alpha}{k})-\alpha\log(\frac{\alpha}{k}))N^2+O(N)} > d_{\alpha N^2}$$

Counting Invariants

We have

$$Z(x) = \frac{1}{\prod_j (1-x^j)^{c_j^p}} (1 + \sum_i c_i^s x^i)$$

with

$$[x^{\alpha N^2}] Z(x) > e^{N^2 \log 2}$$

by the estimate of the Littlewood-Richardson coefficient, and

$$[x^{\alpha N^2}] \frac{1}{\prod_j (1-x^j)^{c_j^p}} < e^{\frac{1}{5}((5+\alpha)\log(1+\frac{\alpha}{5}) - \alpha \log \frac{\alpha}{5}) N^2 + O(N)}$$

Since $\frac{1}{5}((5+\alpha)\log(1+\frac{\alpha}{5}) - \alpha \log \frac{\alpha}{5}) < \log 2$ we must have

$$[x^{\alpha N^2}](1 + \sum_i c_i^s x^i) = e^{cN^2}$$

Generic Secondary

Number of secondary invariants grows as $O(e^{cN^2})$: only possible if almost all secondary invariants have a length of order N^2 - dual to new spacetime backgrounds.

This many background states would give rise to an entropy of

$$S = \log(e^{cN^2}) = \frac{c}{\left(\frac{1}{N^2}\right)} = \frac{c}{G_N}$$

reminiscent of a black hole entropy.

As N is increased the character of an invariant can change - something that was originally secondary can transition to become primary as N is increased. This looks like a purely bosonic analogue of fortuity.

We also expect secondary operators that correspond to solitons - objects like giant gravitons. These secondaries should have a dimension of order N.

Finally since the number of single trace operators with $\leq N$ matrices in the trace grows as e^N and there are only $O(N^2)$ primary invariants, we also expect many short secondaries constructed from $O(1)$ fields. What is their interpretation?

Comment on Primary vs Secondary Invariants

Number of primary invariants $M = (d-1)N^2 + 1$.

Single-trace operators with $\le N$ matrices are generating invariants: can't be eliminated as trace relations only start at $N+1$ matrices in the trace.

Length L is the number of matrices in the trace. Words of length L:

$$X^{a_1} X^{a_2} \times \cdots \times X^{a_L}$$

$$(\text{any of } d) \times (\text{any of } d) \times \cdots\cdots (\text{any of } d)$$

The number of words is: $N_{\text{words}}(L) \;=\; d^L$

This is greater than the number of single trace operators of length L since

$$XXXY \quad XXYX \quad XYXX \quad YXXX \quad \leftrightarrow \quad \mathrm{Tr}(X^3 Y)$$

$$XYXY \quad YXYX \quad \leftrightarrow \quad \mathrm{Tr}(XYXY)$$

$$XXXX \quad \leftrightarrow \quad \mathrm{Tr}(X^4)$$

At **most** L words of length L give the same single trace operator.

Comment on Primary vs Secondary Invariants

Number of primary invariants $M = (d-1)N^2 + 1$.

Single-trace operators with $\leq N$ matrices are generating invariants: can't be eliminated as trace relations only start at $N+1$ matrices in the trace.

Length $L =$ number of matrices in trace. Number of single-trace operators of length L is

$$N_{\text{op}}(L) \approx \frac{d^L}{L} \approx e^{L \log d - \log L}$$

Captures leading behavior, but systematically underestimates the actual count.

For modest value of N, total number of single trace operators of length N ($\sim e^N$) vastly exceeds number of primary invariants ($= (d-1)N^2 + 1$). Although single-trace operators with $\leq N$ matrices are included, tiny fraction are actually primary invariants.

Example: Two matrix model at $N = 20$. Total number of single-trace operators with $\leq N$ matrices is 111,321. Number of primary invariants is $M = N^2 + 1 = 401$.

Small Secondaries

The small secondaries can be decomposed into two types

- **Irreducible secondaries:** Single trace operators with $< N$ matrices in the trace.
- **Reducible secondaries:** Given by product of the irreducible secondaries.

The net result is that we have

$$(S_{\mathrm{irr},1})^{n_1}(S_{\mathrm{irr},2})^{n_2}\cdots(S_{\mathrm{irr,K}})^{n_K}$$

but there is an upper limit for the power n_i.

Clarifying comment: You should not think that above we have abandoned our rule that secondaries only appear linearly. The above operator is obtained when constructing the complete set of secondaries.

Irreducible secondary invariants as q-reducible oscillators: they act like independent creation operators at low excitation levels, but beyond a critical power their action becomes redundant. Their finite range of action distinguishes them sharply from the genuinely free (Fock-space) degrees of freedom generated by the primaries.

Summary

The space of gauge invariant operators that can be constructed in the multi-matrix model quantum mechanics of d matrices is generated from primary and secondary invariants

$$\mathcal{H} = \bigoplus_{B=1}^{N_S} \prod_{A=1}^{N_P} \sum_{\{n_A\}=0}^{\infty} (P_A)^{n_A} S_B$$

The number of primary invariants is $(d-1)N^2+1$. The primary invariants generate a Fock space structure and represent perturbative degrees of freedom.

The number of secondary invariants grows as $\sim e^{N^2}$. We conjecture that black hole microstates are represented as secondary invariants.

As N is increased the character of invariants changes and secondary invariants transition to become primary invariants. This is a bosonic analog of the fortuity mechanism.

The trace relations implies that almost all of the oscillators present in the IR are cut off as we move to the UV: the trace relations implement an enormous reduction in the number of degrees of freedom in the UV of quantum gravity.

Thanks for your attention!