Holomorphic QFT and Chiral Deformations

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Holomorphic QFT is the complex analogue of topological QFT

$$Q_{\mathsf{BRST}} = \bar{\partial} + \cdots$$

Typically, fields are built from Dolbeault complex $\Omega^{0,\bullet}(X)$.

Examples:

- Holomorphic $\beta - \gamma$ system: $\int \beta \bar{\partial} \gamma$
- Holomorphic Chern-Simons theory: $\int \mathcal{A} \bar{\partial} \mathcal{A} + \cdots$
- Kodaira-Spencer gravity
- Holomorphically twisted theory
- $\cdots$

The two-point function (propagator) is related to

$$\bar{\partial}^{-1} = \text{Bochner–Martinelli kernel}$$

On $\mathbb{C}$,

$$\bar{\partial}^{-1} = \frac{1}{z - w}$$

On $\mathbb{C}^n$ ($n > 1$), the Bochner–Martinelli kernel is

$$\bar{\partial}^{-1} = \frac{(n-1)!}{(2\pi i)^n} \sum_{j=1}^{n} (-1)^{j-1} \frac{\bar{z}_j - \bar{w}_j}{|z - w|^{2n}} \, d\bar{z}_1 \wedge \cdots \wedge \widehat{d\bar{z}_j} \wedge \cdots \wedge d\bar{z}_n \wedge dz_1 \wedge \cdots \wedge dz_n$$

Topological QFT's (e.g Chern-Simons) are UV finite. The UV finite property was established by **Kontsevich** and **Axelrod-Singer** using the compactified configuration space.

$$\int_{\mathrm{Conf}_n(X)} \Phi_\Gamma = \int_{\overline{\mathrm{Conf}_n(X)}} \overline{\Phi}_\Gamma$$

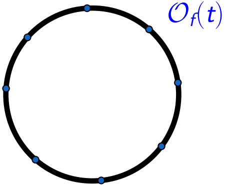
Here

$$\mathrm{Conf}_n(X) = \{(x_1, \cdots, x_n) \in X^n) | x_i \neq x_j, \forall i, j\}$$

Remarkably, holomorphic QFT's are also free of UV divergence! In holomorphic QFT, the Feynman graph integral can not be extended to the compactified configuration space. The UV finite property is by completely different analytic reason.

- In $\dim_{\mathbb{C}} = 1$. This is essentially Cauchy principal value. A regularized integral theory was developed by **L-Zhou**.
- In $\dim_{\mathbb{C}} > 1$
  - **Costello-L** and **Williams**: one-loop is UV finite
  - **Budzik-Gaiotto-Kulp-Wu-Yu**: Laman graphs are UV finite
  - **Minghao Wang**: all graphs in hol QFT are UV finite.
  - **Wang-Yan**: hol QFT on Kahler manifolds are UV finite.

Motivation: Top QM $\Longrightarrow$ Algebraic index



$$\longrightarrow \quad X$$

$$\int_{\mathrm{Conf}^0_{n+1}(S^1)} \langle \mathcal{O}_{f_0}(t_0) \cdots \mathcal{O}_{f_n}(t_n) \rangle$$ [**Grady-L-Li**, **Gui-L-Xu**]

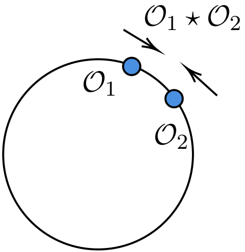
$\Longrightarrow$ Trace map on deformation quantized algebra on X

$\Longrightarrow$ Algebraic Index Theorem : $\mathrm{Tr}(1) = \int_X e^{\frac{1}{\hbar}\omega} \hat{A}(X)$

Goal: chiral elliptic index from $E \to X$.

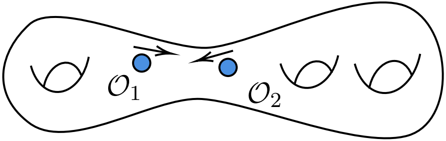
1d TQM	2d Chiral CFT
S^1	Σ
Associative algebra	Vertex operator algebra

Associative product



Operator product expansion

$$\mathcal{O}_1(z)\mathcal{O}_2(w) \sim \sum_n \frac{\mathcal{O}_{1(n)}\mathcal{O}_2(w)}{(z-w)^{n+1}}$$



We are interested in chiral correlations on Riemann surface Σ

$$\langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\cdots\mathcal{O}_n(z_n)\rangle_\Sigma$$

- holomorphic on $\mathrm{Conf}_n(\Sigma) = \{(z_1, \cdots, z_n) \in \Sigma^n | z_i \neq z_j, \forall i, j\}$
- singular with meromorphic poles when $z_i \to z_j$

$$\mathcal{O}_i(z_i)\mathcal{O}_j(z_j) \sim \frac{*}{(z_i - z_j)^{2025}} + \cdots$$

Motivated by TQM/Index, we are led to consider

$$\int_{\Sigma^n} \langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\cdots\mathcal{O}_n(z_n)\rangle_\Sigma \, dVol$$

Unlike the situation in topological field theory, $\langle \mathcal{O}_1(z_1)\cdots\mathcal{O}_n(z_n)\rangle$ is very singular along diagonals and there is no way to extend it to certain compactification of $\mathrm{Conf}_n(\Sigma)$.

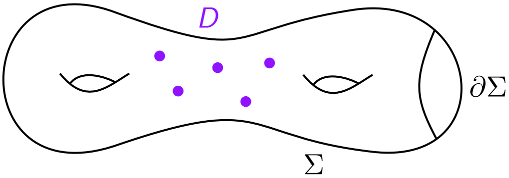
We need to give a precise meaning to the naively divergent integral

$$\int_{\Sigma^n} \Omega$$

where Ω is a differential form on the product Σ^n with arbitrary meromorphic poles along the diagonals.

Regularized integral (L-Zhou 2021)

Let us first consider the integral of a 2-form ω on Σ with meromorphic poles of arbitrary orders along a finite subset $D \subset \Sigma$.



Locally we can write $\omega = \frac{\eta}{z^n}$ where η is smooth 2-form and $n \in \mathbb{Z}$.

We can decompose ω into

$$\omega = \alpha + \partial\beta$$

- α is a 2-form with at most logarithmic pole along D
- β is a $(0,1)$-form with arbitrary order of poles along D
- $\partial = dz\frac{\partial}{\partial z}$ is the holomorphic de Rham

We define the regularized integral

$$\boxed{\fint_{\Sigma} \omega := \int_{\Sigma} \alpha + \int_{\partial\Sigma} \beta}$$

This does not depend on the choice of the decomposition and is equvalent to the Cauchy principal value.

The regularized integral can be viewed as a "homological integration" by the holomorphic de Rham ∂

$$\fint_{\Sigma} \partial(-) = \int_{\partial\Sigma} (-).$$

The $\bar{\partial}$-operator intertwines the residue

$$\fint_{\Sigma} \bar{\partial}(-) = \mathrm{Res}(-).$$

Here

$$\mathrm{Res}_0 \frac{\rho(z,\bar{z})}{z^n} dz = \lim_{\epsilon \to 0} \oint_{|z|=\epsilon} \frac{\rho(z,\bar{z})}{z^n} dz$$

Example:

$$\int_{\mathbb{C}} \frac{d^2 z}{(z-a)(z-b)(z-c)}$$

$$= \frac{\bar{a}}{(a-b)(a-c)} + \frac{\bar{b}}{(b-a)(b-c)} + \frac{\bar{c}}{(c-a)(c-b)}$$

In general, we can define

$$\fint_{\Sigma^n} (-) := \fint_{\Sigma} \fint_{\Sigma} \cdots \fint_{\Sigma} (-)\,.$$

This does not depend on the choice of the ordering (**Fubini** type theorem holds). This gives an intrinsic definition of

$$\fint_{\Sigma^n} \langle \mathcal{O}_1(z_1) \cdots \mathcal{O}_n(z_n) \rangle \, dVol$$

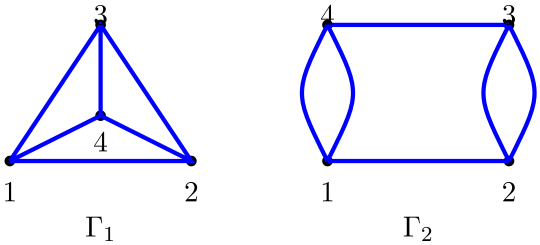
Example: Consider chiral boson on elliptic curve E_τ.

$$\langle \partial\phi(z_1)\partial\phi(z_2)\rangle_{E_\tau} = \widehat{P}(z_1, z_2; \tau, \bar{\tau})$$

Here

$$\widehat{P}(z_1, z_2; \tau, \bar{\tau}) = \wp(z_1 - z_2; \tau) + \frac{\pi^2}{3}\widehat{E}_2(\tau, \bar{\tau})$$
$$\widehat{E}_2(\tau, \bar{\tau}) = E_2(\tau) - \frac{3}{\pi}\frac{1}{\operatorname{im}\tau}$$

$\wp$ is the Weierstrass $\wp$-function, E_2 is the 2nd Eisenstein series.



$$\Phi_{\Gamma_1}(z_1,z_2,z_3,z_4;\tau)=\widehat{P}(z_1-z_2)\widehat{P}(z_2-z_3)\widehat{P}(z_3-z_1)\widehat{P}(z_1-z_4)\widehat{P}(z_2-z_4)\widehat{P}(z_3-z_4)$$

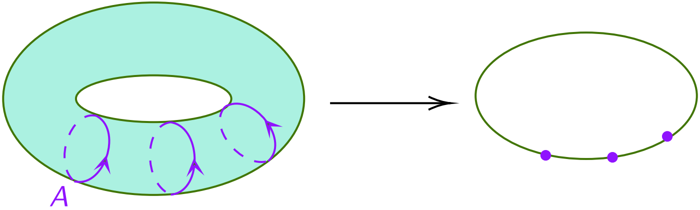
$$\fint_{E_\tau^4}\left(\prod_{i=1}^{4}\frac{d^2z_i}{\operatorname{im}\tau}\right)\Phi_{\Gamma_1}=\frac{(2\pi i)^{12}}{2^{11}\cdot 3^5}(-\widehat{E}_2^6+3\widehat{E}_2^4E_4-3\widehat{E}_2^2E_4^2+E_4^3)$$

$$\Gamma_2=\frac{(2\pi i)^{12}}{2^{10}\cdot 3^7}(-3\widehat{E}_2^6+6\widehat{E}_2^4E_4+4\widehat{E}_2^3E_6-3\widehat{E}_2^2E_4^2-12\widehat{E}_2E_4E_6+4E_4^3+4E_6^2)$$

They are almost holomorphic modular forms.

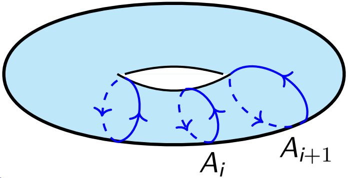
2d → 1d Reduction

In physics, the partition functions on elliptic curves are described by reducing to a quantum mechanical system on S^1.



$$\langle - \rangle_{2d} \longrightarrow \mathrm{Tr}\ \langle \cdots \rangle$$

In 2d we have the *regularized integral* $\fint_E$. In 1d, operators are described by A-cycle $\oint_A$. These two integrals are not exactly the same, but related to each other by *holomorphic limit*.



Theorem (L-Zhou)

$$\fint_{E_\tau^n}\left(\prod_{i=1}^n \frac{d^2 z_i}{\operatorname{im}\tau}\right)\langle \mathcal{O}_1(z_1)\cdots\mathcal{O}_n(z_n)\rangle_E \quad \textit{lies in} \quad \mathcal{O}_{\mathbf{H}}[\frac{1}{\operatorname{im}\tau}]$$

Let $\lim_{\bar\tau\to\infty} : \mathcal{O}_{\mathbf{H}}[\frac{1}{\operatorname{im}\tau}] \to \mathcal{O}_{\mathbf{H}}$ *which sends* $\frac{1}{\operatorname{im}\tau} \to 0$. *Then*

$$\lim_{\bar\tau\to\infty}\fint_{E_\tau^n}\left(\prod_{i=1}^n \frac{d^2 z_i}{\operatorname{im}\tau}\right)\langle \mathcal{O}_1(z_1)\cdots\mathcal{O}_n(z_n)\rangle_E$$
$$= \frac{1}{n!}\sum_{\sigma\in S_n}\int_{A_{\sigma(1)}} dz_1\cdots\int_{A_{\sigma(n)}} dz_n \,\langle \mathcal{O}_1(z_1)\cdots\mathcal{O}_n(z_n)\rangle_E$$

Here $A_1,\cdots,A_n$ be n disjoint A-cycles on E_τ.

This theorem gives a precise relation on reduction of torus to circle

$$\fint_{E_\tau^n} \xrightarrow{\lim\limits_{\bar\tau\to\infty}} \text{Weyl ordered } \oint_A$$

The anti-holomorphic dependence of $\fint_{E^n}$ on the moduli τ is fully described by the following holomorphic anomaly equation

$$\partial_{\mathbb{Y}} \fint_{E^n}(-) = \fint_{E_\tau^n} \partial_{\mathbb{Y}}(-) - \sum_{a,b:\, a<b} \fint_{E_\tau^{n-\{a\}}} \mathrm{Res}_{z_a=z_b}((z_a - z_b)(-))\,.$$

Here $\mathbb{Y} = -\frac{\pi}{\operatorname{im}\tau}$.

Chiral Deformation (after Douglas-Dijkgraaf)

Consider a deformation of chiral boson-$\beta\gamma - bc$-systems by

$$\int_{E_\tau} \mathcal{L} \qquad \text{chiral: only hol derivatives of fields}$$

Define the partition function of the chiral deformed theory

$$\left\langle e^{\frac{1}{\hbar}\int_{E_\tau}\mathcal{L}} \right\rangle_{E_\tau} := \sum_{n=0}^{\infty} \frac{1}{\hbar^n n!} \fint_{E_\tau^n} \langle \mathcal{L}(z_1)\cdots\mathcal{L}(z_n)\rangle_{E_\tau}$$

Theorem [Hou-L-Zhu] The following elliptic trace formula holds

$$\lim_{\bar{\tau}\to\infty} \left\langle e^{\frac{1}{\hbar}\int_{E_\tau}\mathcal{L}} \right\rangle_{E_\tau} = \frac{\mathrm{Tr}\ q^{L_0-\frac{c}{24}} e^{\frac{1}{\hbar}\oint\mathcal{L}}}{\mathrm{Tr}\ q^{L_0-\frac{c}{24}}}$$

The operation $\lim\limits_{\bar{\tau}\to\infty}$ sends

$$\mathbb{C}[\hat{E}_2, E_4, E_6] \quad \Longrightarrow \quad \mathbb{C}[E_2, E_4, E_6]$$

almost holomorphic modular forms $\Longrightarrow$ quasi-modular forms.

Application: The full quantum B-model (quantum BCOV theory as developed in **Costello-L**) on elliptic curves is a chiral deformation

$$S = \int \partial\phi \wedge \bar{\partial}\phi + \sum_{k\geq 0} \int \eta_k \frac{W^{(k+2)}(\partial_z\phi)}{k+2}$$

where

$$W^{(k)}(\partial_z\phi) = (\partial_z\phi)^k + O(\hbar)$$

are the bosonic realization of the $W_{1+\infty}$-algebra. The holomorphic limit $\bar{\tau} \to \infty$ of the generating function of S on the elliptic curve coincides with the Gromov-Witten invariants on the mirror computed by **Dijkgraaf** and **Okounkov-Pandharipande**.

Example: Holomorphic Chern Simons theory

X Calabi-Yau 3-fold, $\mathfrak{g}$ Lie algebra. Fields: $\mathcal{A} \in \Omega^{0,\bullet}(X, \mathfrak{g})[1]$

$$HCS[\mathcal{A}] = \int_X \mathrm{Tr}\left(\frac{1}{2}\mathcal{A} \wedge \bar{\partial}\mathcal{A} + \frac{1}{6}\mathcal{A} \wedge [\mathcal{A}, \mathcal{A}]\right) \wedge \Omega_X$$

- BRST transformation

$$\delta_{HCS}(\mathcal{A}) = \bar{\partial}\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}].$$

- Classical solutions: holomorphic vector bundles.

Unlike ordinary (topological) CS theory, HCS has a huge freedom to deform preserving gauge symmetry. The 1st order deformation is

$$HCS \to HCS + J$$

where J is a local functional which is BRST closed

$$\delta_{HCS} J = 0.$$

They are parametrized by the BRST cohomology

$$\mathrm{Def}(HCS) = H^{\bullet}(\mathcal{O}_{loc}, \delta_{HCS})$$

Example: Consider a tensor field (called a polyvector field)

$$\mu = \mu^{i_1 \cdots i_k}_{\bar{j}_1 \cdots \bar{j}_m} \in \mathrm{PV}^{k,m}(X)$$

which is totally skew-symmetric in i's and in j's. Then

$$\int_X \mu^{i_1 \cdots i_k} \,\mathrm{Tr}(\mathcal{A} \wedge \partial_{z^{i_1}} \mathcal{A} \wedge \cdots \wedge \partial_{z^{i_k}} \mathcal{A}) \wedge \Omega_X$$

gives a 1st-order deformation if μ is divergence free.

Costello-L: 1st order deformations of HCS at $N \to \infty$ are

$$(\mathrm{PV}(X)[[t]], Q = \bar{\partial} + t\partial)$$

which is the Kodaira-Spencer theory with gravitational descendant (we still call this BCOV theory). This follows from a classical theorem of **Loday-Quillen-Tsygan** which computes the Lie algebra (co)homology of at $N = \infty$.

We discover Kodaira-Spencer gravity by chiral deformations of HCS in the large N limit! In fact, full quantum dynamics are recovered.

Thanks!