Stable envelope for critical loci

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based on work to appear with Yalong Cao, Andrei Okounkov, and Zijun Zhou

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4. It will naturally follow from construction that

$$\mathsf{Y}_\mu(Q, W) \curvearrowright \bigoplus_{\mathbf{v}} H_{\mathrm{eq,crit}}(\mathcal{M}(\mathbf{v}, \underbrace{\mathbf{d}_{\mathrm{in}}, \mathbf{d}_{\mathrm{out}}}_{\mathbf{d}_{\mathrm{out}}-\mathbf{d}_{\mathrm{in}}=\mu}), W^{\mathrm{fr}}) \rightsquigarrow \text{Crystals}$$

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Then the $\mathbb{C}^*$-fixed points decomposes into disjoint union of products

$$\mathcal{N}(\mathbf{v}, \mathbf{d})^{\mathbb{C}^*} = \bigsqcup_{\mathbf{v}' + \mathbf{v}'' = \mathbf{v}} \mathcal{N}(\mathbf{v}', \mathbf{d}') \times \mathcal{N}(\mathbf{v}'', \mathbf{d}'')$$

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- E.g. F can be $\mathbb{C}^*_\hbar$ that scales cotangent fiber in $T^*\mathrm{Rep}$.

Proposition [MO, 2012]

There exists a *unique* $H_{\mathrm{eq}}(\mathrm{pt})$-linear map (called *stable envelope*)

$$\mathrm{Stab}: H_{\mathrm{eq}}(X^{\mathbb{C}^*}) \to H_{\mathrm{eq}}(X)$$

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3. $\forall \delta \geq 0$,

$$\deg_{\mathbb{C}^*} \mathrm{Stab}(\gamma)\Big|_{\mathcal{N}(\mathbf{v}'-\delta,\mathbf{d}') \times \mathcal{N}(\mathbf{v}''+\delta,\mathbf{d}'')} < \mathrm{rk} N^{\mathrm{repelling}}_{\mathcal{N}(\mathbf{v}'-\delta,\mathbf{d}') \times \mathcal{N}(\mathbf{v}''+\delta,\mathbf{d}'')/X}$$

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We define the opposite stable envelope to be

$$\mathrm{Stab}_{-} : \mathcal{H}_{\mathbf{d}'} \otimes \mathcal{H}_{\mathbf{d}''} \xrightarrow{\text{swap}} \mathcal{H}_{\mathbf{d}''} \otimes \mathcal{H}_{\mathbf{d}'} \xrightarrow[\mathbf{d}=\mathbf{d}''+\mathbf{d}']{\text{Stab for}} \mathcal{H}_{\mathbf{d}} \xrightarrow[\cong]{u \mapsto -u} \mathcal{H}_{\mathbf{d}}$$

where $H_{\mathbb{C}^*}(\mathrm{pt}) = \mathbb{C}[u]$.

R-matrix and YBE

Define the *R-matrix*

$$R(u) := (\mathrm{Stab}_-)^{-1} \circ \mathrm{Stab} \in \mathrm{End}(\mathcal{H}_{\mathbf{d}'} \otimes \mathcal{H}_{\mathbf{d}''}(u)).$$

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Consider $\mathcal{H}_{\mathbf{d}_1} \otimes \mathcal{H}_{\mathbf{d}_2} \otimes \mathcal{H}_{\mathbf{d}_3}$, and let

$$R_{12} := (\text{R-matrix for } \mathcal{H}_{\mathbf{d}_1} \otimes \mathcal{H}_{\mathbf{d}_2}) \otimes \mathrm{Id}_{\mathcal{H}_{\mathbf{d}_3}}$$

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Theorem [MO 2012]

Yang-Baxter equation (YBE) holds:

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u).$$

FRT formalism

Given a colletion of vector spaces $\{F_i\}_{i\in I}$ with endomorphisms

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The complete definition is wordy, roughly speaking, Y is gen. by the matrix coeff. in the $u \to \infty$ expansion of

$$R_{i_0,i_n}(u-a_n)\cdots R_{i_0,i_1}(u-a_1), \quad \text{for all } i_0 \in I.$$

Maulik-Okounkov Yangian

Definition

Given a quiver Q, define the Maulik-Okounkov Yangian $\mathsf{Y}_Q^{\mathrm{MO}}$ to be algebra obtained from applying FRT formalism to the vector spaces $\{\mathcal{H}_{\mathbf{d}}\}_{\mathbf{d}\in\mathbb{Z}_{\geq 0}^{Q_0}}$ with R-matrices constructed from stable envelopes as above.

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For the Jordan quiver (one node with one self-loop), $\mathrm{Y}_Q^{\mathrm{MO}} \cong \mathbf{SH^c}$, the $N \to \infty$ limit of spherical degenerate DAHA of type $\mathfrak{gl}_N$ defined in [Schiffmann-Vasserot 2012]. $\mathbf{SH^c} \cong Y(\widehat{\mathfrak{gl}}_1) \otimes Y(\mathfrak{gl}_1)$.

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- Suppose F is a flavour symmetry such that W^{fr} is F-invariant, we define state space to be direct sum of equivariant critical cohomologies:

$$\mathcal{H}_{\underline{\mathbf{d}}}^{W^{\mathrm{fr}}}:=\bigoplus_{\mathbf{v}}H_{\mathsf{F},\mathrm{crit}}(\mathcal{M}(\mathbf{v},\underline{\mathbf{d}}),W^{\mathrm{fr}})$$

Critical stable envelopes

Proposition [Cao-Okounkov-Z.-Zhou, 2025]

There exists a *unique* $H_{\mathrm{eq}}(\mathrm{pt})$-linear map (called *critical stable envelope*)

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with the following properties:$\forall \gamma \in H_{\mathsf{F},\mathrm{crit}}(\mathcal{M}(\mathbf{v}', \underline{\mathbf{d}}') \times \mathcal{M}(\mathbf{v}'', \underline{\mathbf{d}}''), W^{\mathrm{fr}})$

1. $\mathrm{Stab}(\gamma)$ is supported on $\bigcup_{\delta \geq 0} \mathrm{Attr}(\mathcal{M}(\mathbf{v}' - \delta, \underline{\mathbf{d}}') \times \mathcal{M}(\mathbf{v}'' + \delta, \underline{\mathbf{d}}''))$
2. $\mathrm{Stab}(\gamma)\Big|_{\mathcal{M}(\mathbf{v}', \underline{\mathbf{d}}') \times \mathcal{M}(\mathbf{v}'', \underline{\mathbf{d}}'')} = \mathrm{Eul}_{\mathrm{eq}}\left(N^{\mathrm{repelling}}_{\mathcal{M}(\mathbf{v}', \underline{\mathbf{d}}') \times \mathcal{M}(\mathbf{v}'', \underline{\mathbf{d}}'')/X}\right) \cdot \gamma$
3. $\forall \delta \geq 0$,
   $$\deg_{\mathbb{C}^*} \mathrm{Stab}(\gamma)\Big|_{\mathcal{M}(\mathbf{v}'-\delta, \underline{\mathbf{d}}') \times \mathcal{M}(\mathbf{v}''+\delta, \underline{\mathbf{d}}'')} < \mathrm{rk} N^{\mathrm{repelling}}_{\mathcal{M}(\mathbf{v}'-\delta, \underline{\mathbf{d}}') \times \mathcal{M}(\mathbf{v}''+\delta, \underline{\mathbf{d}}'')/X}$$

Collect stable envelopes for all **v**, and we get a map

$$\mathrm{Stab} : \mathcal{H}^{W'}_{\underline{\mathbf{d}}'} \otimes \mathcal{H}^{W''}_{\underline{\mathbf{d}}''} \to \mathcal{H}^{W^{\mathrm{fr}}}_{\underline{\mathbf{d}}}.$$

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We define the opposite stable envelope to be

$$\mathrm{Stab}_- : \mathcal{H}^{W'}_{\underline{\mathbf{d}}'} \otimes \mathcal{H}^{W''}_{\underline{\mathbf{d}}''} \xrightarrow{\mathsf{swap}} \mathcal{H}^{W''}_{\underline{\mathbf{d}}''} \otimes \mathcal{H}^{W'}_{\underline{\mathbf{d}}'} \xrightarrow{\mathsf{Stab}} \mathcal{H}^{W^{\mathrm{fr}}}_{\underline{\mathbf{d}}} \xrightarrow[\cong]{u \mapsto -u} \mathcal{H}^{W^{\mathrm{fr}}}_{\underline{\mathbf{d}}}$$

where $H_{\mathbb{C}^*}(\mathrm{pt}) = \mathbb{C}[u]$.

R-matrix and YBE

Define the *R-matrix*

$$R(u) := (\mathrm{Stab}_{-})^{-1} \circ \mathrm{Stab} \in \mathrm{End}\left(\mathcal{H}^{W'}_{\underline{\mathbf{d}}'} \otimes \mathcal{H}^{W''}_{\underline{\mathbf{d}}''}(u)\right).$$

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Consider $\mathcal{H}^{W_1}_{\underline{\mathbf{d}_1}} \otimes \mathcal{H}^{W_2}_{\underline{\mathbf{d}_2}} \otimes \mathcal{H}^{W_3}_{\underline{\mathbf{d}_3}}$, and let

$$R_{12} := (\text{R-matrix for } \mathcal{H}^{W_1}_{\underline{\mathbf{d}_1}} \otimes \mathcal{H}^{W_2}_{\underline{\mathbf{d}_2}}) \otimes \mathrm{Id}_{\mathcal{H}^{W_3}_{\underline{\mathbf{d}_3}}}$$

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Similarly for other R_{ij}.

Theorem [COZZ 2025]

Assume that Q is symmetric ($\# i \to j = \# j \to i$) with anti-dominant framings $\underline{\mathbf{d}_i}$ ($\mathbf{d}_{i,\mathrm{out}} \leq \mathbf{d}_{i,\mathrm{in}}$) for $i \in \{1, 2, 3\}$, then the Yang-Baxter equation (YBE) holds:

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u).$$

The key step of the proof of the Yang-Baxter equation is the following.

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H_{\mathrm{eq,crit}}(\mathfrak{M}(\mathbf{v}', \underline{\mathbf{d}}'), W') \otimes H_{\mathrm{eq,crit}}(\mathfrak{M}(\mathbf{v}'', \underline{\mathbf{d}}''), W'') \arrow[r, "\mathrm{CoHA}"', "\text{[KS]}"] & H_{\mathrm{eq,crit}}(\mathfrak{M}(\mathbf{v}, \underline{\mathbf{d}}), W^{\mathrm{fr}}) \\
H_{\mathrm{eq,crit}}(\mathcal{M}(\mathbf{v}', \underline{\mathbf{d}}'), W') \otimes H_{\mathrm{eq,crit}}(\mathcal{M}(\mathbf{v}'', \underline{\mathbf{d}}''), W'') \arrow[r, "\mathrm{Stab}", "\text{[COZZ]}"'] \arrow[u, "\Psi\otimes\Psi"] & H_{\mathrm{eq,crit}}(\mathcal{M}(\mathbf{v}, \underline{\mathbf{d}}), W^{\mathrm{fr}}) \arrow[u, "\Psi"']
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Using the compatibility between Stab and CoHA, one can show that the proof of the Yang-Baxter equation boils down to the **associativity** of CoHA:

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\mathfrak{H}^{W_1}_{\mathbf{v}_1,\underline{\mathbf{d}_1}} \otimes \mathfrak{H}^{W_{23}}_{\mathbf{v}_2+\mathbf{v}_3,\underline{\mathbf{d}_2}+\underline{\mathbf{d}_3}} \arrow[dr, "\mathfrak{m}"] & & \mathfrak{H}^{W_{12}}_{\mathbf{v}_1+\mathbf{v}_2,\underline{\mathbf{d}_1}+\underline{\mathbf{d}_2}} \otimes \mathfrak{H}^{W_3}_{\mathbf{v}_3,\underline{\mathbf{d}_3}} \arrow[dl, "\mathfrak{m}"'] \\
 & \mathfrak{H}^{W_{123}}_{\mathbf{v}_1+\mathbf{v}_2+\mathbf{v}_3,\underline{\mathbf{d}_1}+\underline{\mathbf{d}_2}+\underline{\mathbf{d}_3}} &
\end{tikzcd}
$$

where $\mathfrak{H}^{W^{\mathrm{fr}}}_{\mathbf{v},\underline{\mathbf{d}}} := H_{\mathrm{eq,crit}}(\mathfrak{M}(\mathbf{v},\underline{\mathbf{d}}), W^{\mathrm{fr}})$.

Comparison with MO's stable envelopes

Suppose that Q is a quiver, define the **doubled** quiver $\overline{Q}$ by adding an opposite arrow for each arrow of Q.

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- Given a symmetric framing $\underline{\mathbf{d}}$ ($\mathbf{d}_{\mathrm{in}} = \mathbf{d}_{\mathrm{out}}$), we define the **canonical framed cubic potential**

$$W^{\mathrm{fr}}_{\mathrm{can}} = W_{\mathrm{can}} + \sum_{i\in Q_0} \varepsilon_i A_i B_i$$

  for A_i=incoming framing at i, B_i=outgoing framing at i.

Proposition (Dimensional Reduction) [Davison, 2013]

Let $\mathbf{d}_{\mathrm{in}} = \mathbf{d}_{\mathrm{out}} = \mathbf{d}$, then there is a natural isomorphism

$$\mathrm{dr} : H_{\mathrm{eq,crit}}(\mathcal{M}_{\widetilde{Q}}(\mathbf{v}, \underline{\mathbf{d}}), W^{\mathrm{fr}}_{\mathrm{can}}) \cong H_{\mathrm{eq}}(\mathcal{N}_Q(\mathbf{v}, \mathbf{d}))$$

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H_{\mathrm{eq}}(\mathcal{N}_Q(\mathbf{v}', \mathbf{d}')) \otimes H_{\mathrm{eq}}(\mathcal{N}_Q(\mathbf{v}'', \mathbf{d}'')) \arrow[r, "\mathrm{Stab}\cdot(-1)^{\sharp}", "\text{[MO]}"'] & H_{\mathrm{eq}}(\mathcal{N}_Q(\mathbf{v}, \mathbf{d}))
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Table of Contents

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Remark

It follows from construction that there is family of natural alg. hom. $\Delta_{\mu,\mu'} : \mathsf{Y}_{\mu+\mu'} \to \mathsf{Y}_\mu \otimes \mathsf{Y}_{\mu'}$ which is coassociative:

$$(\mathrm{id} \otimes \Delta_{\mu_2,\mu_3}) \circ \Delta_{\mu_1,\mu_2+\mu_3} = (\Delta_{\mu_1,\mu_2} \otimes \mathrm{id}) \circ \Delta_{\mu_1+\mu_2,\mu_3}$$

The algebra $\mathsf{Y}_\mu(Q, W)$

Given a symmetric quiver Q with potential W, let

$$\{H_\alpha\} := \left\{ \mathcal{H}_{\underline{\mathbf{d}}}^{W^{\mathrm{fr}}} \;\middle|\; \mathbf{d}_{\mathrm{out}} \leq \mathbf{d}_{\mathrm{in}},\ W^{\mathrm{fr}} \text{ is an extension of } W \text{ to } \underline{\mathbf{d}}\text{-framed reps} \right\}$$

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Let $\mu(\underline{\mathbf{d}}, W^{\mathrm{fr}}) := \mathbf{d}_{\mathrm{out}} - \mathbf{d}_{\mathrm{in}} \in \mathbb{Z}_{\le 0}^{Q_0}$, and define

$$\{F_i\} := \left\{ \mathcal{H}_{\underline{\mathbf{d}}}^{W^{\mathrm{fr}}} \text{ in the above} \;\middle|\; \mathbf{d}_{\mathrm{out}} = \mathbf{d}_{\mathrm{in}} \right\}$$

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Definition

For $\mu \in \mathbb{Z}_{\leq 0}^{Q_0}$, define the algebra $\mathsf{Y}_\mu(Q, W)$ to be algebra obtained from applying the aforementioned modified FRT formalism to the vector spaces $\{F_i\}$ and $\{H_\alpha\}$ with R-matrices constructed from critical stable envelopes.

Example: trivial quiver, trivial potential

$$Q : \bigcirc \qquad W = 0$$

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Proposition [COZZ 2025]

$\mathsf{Y}_\mu(Q, W) \cong Y_\mu(\mathfrak{gl}_{1|1})$, the μ-shifted Yangian of $\mathfrak{gl}_{1|1}$.

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To define $Y_\mu(\mathfrak{gl}_{1|1})$, let

$$R(z) = z\,\mathrm{Id} + \hbar\, P \in \mathrm{End}(\mathbb{C}^{1|1} \otimes \mathbb{C}^{1|1})[z, \hbar],$$

where $P(a \otimes b) = (-1)^{|a|\cdot|b|} b \otimes a$ is the super permutation operator.

Definition

For $\mu \leq 0$, $Y_\mu(\mathfrak{gl}_{1|1})$ is the $\mathbb{C}[\hbar]$-algebra generated by $\{e_n, f_n, g_n, h_m\}_{n\in\mathbb{Z}_{\geq 0}, m\in\mathbb{Z}_{\geq -\mu}}$, subject to relations

$$R(u)\, T_1(u+v)\, T_2(v) = T_2(v)\, T_1(u+v)\, R(u),$$

where

$$T_i(z) = \begin{pmatrix} 1 & 0 \\ f(z) & 1 \end{pmatrix} \begin{pmatrix} g(z) & 0 \\ 0 & g(z)h(z) \end{pmatrix} \begin{pmatrix} 1 & e(z) \\ 0 & 1 \end{pmatrix} \in \mathrm{End}(\mathbb{C}^2_i)\otimes Y_\mu(\mathfrak{gl}_2)((z^{-1}))$$

$$e(z) = \sum_{n\geq 0} e_n z^{-n-1},\ f(z) = \sum_{n\geq 0} f_n z^{-n-1},$$

$$g(z) = 1 + \sum_{n\geq 0} g_n z^{-n-1},\ h(z) = z^\mu + \sum_{m\geq -\mu} h_m z^{-m-1}$$

As a corollary, we see that $\mathsf{Y}_{-m}(\mathfrak{gl}_{1|1})$ acts on cohomology of

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$$R(u,t_1,t_2) = \frac{1}{u+t_1} \begin{pmatrix} u+t_1 & & & \\ & u & t_2 & \\ & t_1 & u+t_1-t_2 & \\ & & & u-t_2 \end{pmatrix}.$$

It satisfies super YBE

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Special case: $t_1 = t_2 = \hbar$, $R(u, \hbar) = \frac{u+\hbar P}{u+\hbar}$ = fundamental R-matrix.

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- $n = 1, m = 1$. Let $\mathbb{C}^*_\hbar$ act on the out-going arrow with equiv. var. $\hbar$,

$$L(u,\hbar) = \frac{1}{u+\hbar}\begin{pmatrix} u+\hbar & & & \\ & u & 1 & \\ & -\hbar & 1 & \\ & & & 1 \end{pmatrix}.$$

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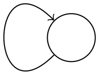
Regarding the second $\mathbb{C}^{1|1}$ in $\mathbb{C}^{1|1} \otimes \mathbb{C}^{1|1}$ as a Clifford module:
$\psi^* |\uparrow\rangle = |\downarrow\rangle, \quad \psi |\downarrow\rangle = |\uparrow\rangle$, where $\{\psi, \psi^*\} = 1$, then

$$L(u,\hbar) = \frac{1}{u+\hbar} \begin{pmatrix} u+\hbar\psi\psi^* & \psi^* \\ -\hbar\psi & 1 \end{pmatrix}.$$

Example: Jordan quiver, $W = 0$

$Q :$  $\quad W = 0$

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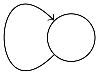
Q :  $W = 0$

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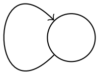
Q :  $W = 0$

Proposition [COZZ 2025]

$\mathsf{Y}_\mu(Q, W = 0) \cong Y_\mu(\mathfrak{gl}_2)$, the μ-shifted Yangian of $\mathfrak{gl}_2$.

To define $Y_\mu(\mathfrak{gl}_2)$, let

$$R(z) = z\,\mathrm{Id} + \hbar\, P \in \mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2)[z, \hbar],$$

where $P(a \otimes b) = b \otimes a$ is the permutation operator.

Definition [equiv. to Frassek-Pestun-Tsymbaliuk 2020]

For $\mu \le 0$, $Y_\mu(\mathfrak{gl}_2)$ is the $\mathbb{C}[\hbar]$-algebra generated by $\{e_n, f_n, g_n, h_m\}_{n\in\mathbb{Z}_{\ge 0}, m\in\mathbb{Z}_{\ge -\mu}}$, subject to relations

$$R(u)\,T_1(u+v)\,T_2(v) = T_2(v)\,T_1(u+v)\,R(u),$$

where

$$T_i(z) = \begin{pmatrix} 1 & 0 \\ f(z) & 1 \end{pmatrix} \begin{pmatrix} g(z) & 0 \\ 0 & g(z)h(z) \end{pmatrix} \begin{pmatrix} 1 & e(z) \\ 0 & 1 \end{pmatrix} \in \mathrm{End}(\mathbb{C}^2_i)\otimes Y_\mu(\mathfrak{gl}_2)(\!(z^{-1})\!)$$

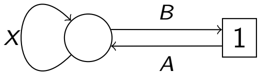
$$e(z) = \sum_{n\ge 0} e_n z^{-n-1},\ f(z) = \sum_{n\ge 0} f_n z^{-n-1},$$

$$g(z) = 1 + \sum_{n\ge 0} g_n z^{-n-1},\ h(z) = z^\mu + \sum_{m\ge -\mu} h_m z^{-m-1}$$

As a corollary, we see that $\mathsf{Y}_\mu(\mathfrak{gl}_2)$ acts on any state space $\mathcal{H}^W_{\underline{\mathbf{d}}}$ with $\mathbf{d}_{\mathrm{out}} - \mathbf{d}_{\mathrm{in}} = \mu$ and a $\underline{\mathbf{d}}$-framed potential W that $W|_{\text{unframed quiver}} = 0$.

As a corollary, we see that $Y_\mu(\mathfrak{gl}_2)$ acts on any state space $\mathcal{H}^W_{\underline{\mathbf{d}}}$ with $\mathbf{d}_{\text{out}} - \mathbf{d}_{\text{in}} = \mu$ and a $\underline{\mathbf{d}}$-framed potential W that $W|_{\text{unframed quiver}} = 0$.

- (Kirillov-Reshetikhin module) $Y_0(\mathfrak{gl}_2)$ acts on $H_{\mathrm{eq,crit}}$ of



$$W = BX^N A$$

KR module $\cong S^N\mathbb{C}^2$.

As a corollary, we see that $\mathsf{Y}_\mu(\mathfrak{gl}_2)$ acts on any state space $\mathcal{H}^W_{\underline{\mathbf{d}}}$ with $\mathbf{d}_{\mathrm{out}} - \mathbf{d}_{\mathrm{in}} = \mu$ and a $\underline{\mathbf{d}}$-framed potential W that $W|_{\text{unframed quiver}} = 0$.

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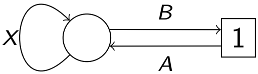
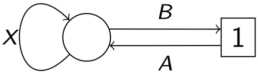


$W = BX^NA$

KR module $\cong S^N\mathbb{C}^2$.

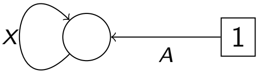
- (Dual Verma module) $\mathsf{Y}_0(\mathfrak{gl}_2)$ acts on $H_{\mathrm{eq,crit}}$ of



$W = 0$

dual Verma module $\cong \mathbb{C}[z]$.

- (Prefundamental module) $\mathsf{Y}_{-1}(\mathfrak{gl}_2)$ acts on $H_{\text{eq,crit}}$ of



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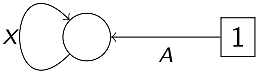

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as a vector space, prefundamental $\cong$ dual Verma $\cong \mathbb{C}[z]$.

Remark

All the R-matrices between the above critical cohomologies are explicitly computable, and agree with the known algebraic R-matrices for those modules.

Example: tripled Jordan quiver, W_{can}

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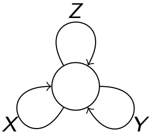

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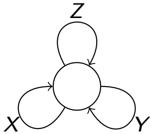

$$W_{\mathrm{can}} = Z[X, Y].$$

Proposition [COZZ 2025]

For every $\mu \in \mathbb{Z}_{\leq 0}$, there exists a natural *surjective* algebra homomorphism

$$\varrho_\mu : \mathsf{Y}_\mu(\widehat{\mathfrak{gl}}_1) \twoheadrightarrow \mathsf{Y}_\mu(Q, W_{\mathrm{can}}).$$

Moreover, ϱ_0 is an isomorphism.

For an integer $\mu \in \mathbb{Z}$, the μ-shifted affine Yangian of $\mathfrak{gl}_1$, $Y_\mu(\widehat{\mathfrak{gl}}_1)$ is defined to be the $\mathbb{C}[\hbar_1, \hbar_2, \hbar_3]/(\hbar_1 + \hbar_2 + \hbar_3)$ algebra generated by $\{e_j, f_j, g_j, c_j\}_{j\in\mathbb{Z}_{\geqslant 0}}$ subject to relations:

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$$
\begin{aligned}
&\text{(Y0)} && [g_i, g_j] = 0, \quad c_j \text{ is central},\\
&\text{(Y1)} && [e_{i+3}, e_j] - 3[e_{i+2}, e_{j+1}] + 3[e_{i+1}, e_{j+2}] - [e_i, e_{j+3}] + \sigma_2([e_{i+1}, e_j] - [e_i, e_{j+1}]) = \sigma_3\{e_i, e_j\},\\
&\text{(Y2)} && [f_{i+3}, f_j] - 3[f_{i+2}, f_{j+1}] + 3[f_{i+1}, f_{j+2}] - [f_i, f_{j+3}] + \sigma_2([f_{i+1}, f_j] - [f_i, f_{j+1}]) = -\sigma_3\{f_i, f_j\},\\
&\text{(Y3)} && [e_i, f_j] = \sigma_3 h_{i+j},\\
&\text{(Y4)} && [g_i, e_j] = e_{i+j},\\
&\text{(Y5)} && [g_i, f_j] = -f_{i+j},\\
&\text{(Y6)} && \mathrm{Sym}_{\mathfrak{S}_3}[e_{i_1}, [e_{i_2}, e_{i_3+1}]] = 0, \quad \mathrm{Sym}_{\mathfrak{S}_3}[f_{i_1}, [f_{i_2}, f_{i_3+1}]] = 0,
\end{aligned}
$$

where $\sigma_2 = \hbar_1\hbar_2 + \hbar_2\hbar_3 + \hbar_3\hbar_1$, $\sigma_3 = \hbar_1\hbar_2\hbar_3$, and the RHS of (Y3) is given by

$$1+\sum_{n\geqslant 0} h_{n-\mu} z^{-n-1} = \exp\left(\sum_{j\geqslant 1} \frac{c_{j-1}}{j\, z^j} + \sum_{k\geqslant 0} g_k \psi_k(z)\right),$$

where $(\psi_k(z))_{k\in\mathbb{Z}_{\geqslant 0}}$ is a sequence of functions such that $\exp\left(\sum_{k\geqslant 0} a^k \psi_k(z)\right) = \prod_{s=1}^3 \frac{z+\hbar_s-a}{z-\hbar_s-a}$ for all $a \in \mathbb{C}$, and we set $h_{-\mu-1} = 1$ and $h_j = 0$ for $j < -\mu - 1$.

As a corollary, we see that $\mathsf{Y}_\mu(\widehat{\mathfrak{gl}}_1)$ acts on any state space $\mathcal{H}^W_{\underline{\mathbf{d}}}$ with $\mathbf{d}_{\mathrm{out}} - \mathbf{d}_{\mathrm{in}} = \mu$ and a $\underline{\mathbf{d}}$-framed potential W that extends W_{can}.

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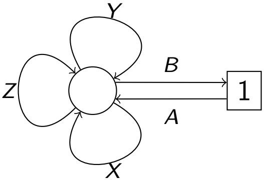

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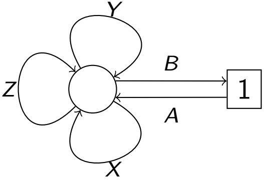

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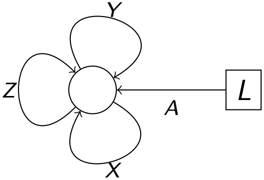
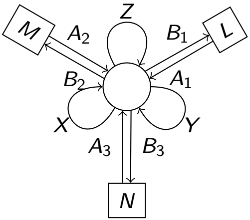
- ($\bigotimes$ of MacMahon modules) $\mathsf{Y}_{-L}(\widehat{\mathfrak{gl}}_1)$ acts on $H_{\mathrm{eq,crit}}$ of



$$, \quad W = W_{\mathrm{can}} = Z[X, Y]$$

- (Free-field realization of Gaiotto-Rapčák's Y-algebra) $\mathrm{Y}_0(\widehat{\mathfrak{gl}}_1)$ acts on $H_{\mathrm{eq,crit}}$ of



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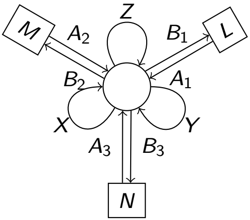

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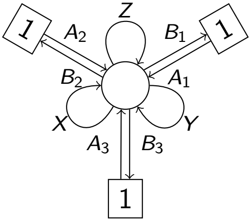

, $W = Z[X,Y] + B_1 X^L A_1 + B_2 Y^M A_2 + B_3 Z^N A_3$

Remark

All the above examples of modules of $Y_\mu(\widehat{\mathfrak{gl}}_1)$ are irreducible for generic equiv. parameters. In fact, we gave geometric criterion for the irreducibility of modules for $\mathsf{Y}_\mu(Q, W)$ of a general symmetric (Q, W).

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For all $\mu \in \mathbb{Z}_{\leq 0}$, $\varrho_\mu : Y_\mu(\widehat{\mathfrak{gl}}_1) \to \mathsf{Y}_\mu(\text{Tripled Jordan}, W_{\text{can}})$ is an isomorphism.

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It follows from generators and relations that $Y_0(\widehat{\mathfrak{gl}}_1) \cong \mathbf{SH^c}$, the $N \to \infty$ limit of spherical degenerate DAHA of type $\mathfrak{gl}_N$ defined in [Schiffmann-Vasserot 2012].

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In fact, we have the following general result.

Compare $\mathsf{Y}_0(\widetilde{Q}, W_{\mathrm{can}})$ with $\mathsf{Y}_Q^{\mathrm{MO}}$

Theorem [COZZ 2025]

For any quiver Q, let $\widetilde{Q}$ be its tripled quiver, $W_{\mathrm{can}} = \sum_i \varepsilon_i \mu_i$ be the canonical cubic potential. Then there is an algebra isomorphism

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such that the dimensional reduction isomorphism

$$\mathrm{dr} : H_{\mathrm{eq,crit}}(\mathcal{M}_{\widetilde{Q}}(\mathbf{v}, \underline{\mathbf{d}}), W^{\mathrm{fr}}_{\mathrm{can}}) \cong H_{\mathrm{eq}}(\mathcal{N}_Q(\mathbf{v}, \mathbf{d})) \quad (\mathbf{d}_{\mathrm{in}} = \mathbf{d}_{\mathrm{out}} = \mathbf{d})$$

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Corollary

$\mathsf{Y}_Q^{\mathrm{MO}}$ acts on $H_{\mathrm{eq,crit}}(\mathcal{M}_{\widetilde{Q}}(\mathbf{v}, \underline{\mathbf{d}}), W)$ with $\mathbf{d}_{\mathrm{in}} = \mathbf{d}_{\mathrm{out}}$ but W can be an arbitrary $\underline{\mathbf{d}}$-framed potential that extends W_{can}.

For example, take $Q =$ affine A_{n-1} type quiver, then $\mathsf{Y}_Q^{\mathrm{MO}}$ acts on the equivariant BM homology of moduli space of parabolic sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$ (affine Laumon space), which has a realization as a chainsaw quiver variety.

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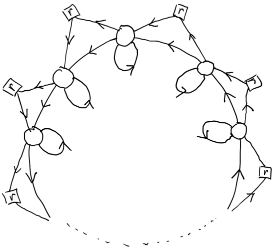

H_{eq}(chainsaw quiver variety) is dimensional reduction of a critical cohomology of form $H_{\mathrm{eq,crit}}(\mathcal{M}_{\widetilde{Q}}(\mathbf{v}, \underline{\mathbf{d}}), W)$.

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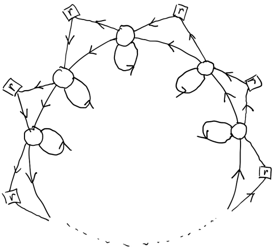

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Conjecture (Parabolic AGT Correspondence)

The above action factors through the rectangular $\mathcal{W}$-algebra $\mathcal{W}(\mathfrak{gl}_{nr}, \text{nilp. of type } (rr \cdots r))$

Filtration on $\mathrm{Y}_0(Q, W)$

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Theorem [COZZ 2025]

There is a natural filtration on $\mathsf{Y}_0(Q, W)$:

$$0 = F_{-1}\mathsf{Y}_0(Q, W) \subset F_0\mathsf{Y}_0(Q, W) \subset F_1\mathsf{Y}_0(Q, W) \subset \cdots$$

and the associated graded is isomorphic to the univ. enveloping algebra

$$\mathrm{gr}\, \mathsf{Y}_0(Q, W) \cong U(\mathfrak{g}_{Q,W}[z])$$

where $\mathfrak{g}_{Q,W}$ is a Lie superalgebra determined by (Q, W).

Structure of $\mathfrak{g}_{Q,W}$

We have the following facts for $\mathfrak{g}_{Q,W}$.

1. $\mathfrak{g}_{Q,W}$ is $\mathbb{Z}^{Q_0}$-graded:

$$\mathfrak{g}_{Q,W}=\bigoplus_{\alpha\in\mathbb{Z}^{Q_0}}\mathfrak{g}_\alpha, \quad [\mathfrak{g}_\alpha,\mathfrak{g}_\beta]\subseteq\mathfrak{g}_{\alpha+\beta}\,.$$

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2. $\mathfrak{g}_0$ is a maximal super-commutative subalgebra, which decomposes as

$$\mathfrak{g}_0=\mathfrak{V}\oplus\mathfrak{D}, \quad \mathfrak{V}=\bigoplus_{i\in Q_0}\mathbb{K}\cdot\mathbb{V}_i, \quad \mathfrak{D}=\bigoplus_{i\in Q_0}\mathbb{K}\cdot\mathbb{D}_i, \quad (\mathbb{K}=\mathrm{Frac}\, H_{\mathrm{eq}}(\mathrm{pt})),$$

$$\mathbb{V}_i\curvearrowright H_{\mathrm{eq,crit}}(\mathcal{M}(\mathbf{v},\underline{\mathbf{d}}),W^{\mathrm{fr}}) \text{ by scalar } \mathbf{v}_i,$$

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4. There exists a non-degenerate super symmetric invariant bilinear form

$$(\cdot,\cdot)_{\mathfrak{g}}:\mathfrak{g}_{Q,W}\times\mathfrak{g}_{Q,W}\to\mathbb{K}$$

With respect to this form, we have $\mathfrak{g}_{-\eta}\cong\mathfrak{g}_\eta^\vee$.

5. The restriction of $(\cdot,\cdot)_{\mathfrak{g}}$ to $\mathfrak{g}_0$ yields a non-degenerate symmetric bilinear form on $\mathfrak{g}_0 = \mathfrak{V} \oplus \mathfrak{D}$, which is explicitly given by the matrix

$$\begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & \boldsymbol{Q} \end{pmatrix} \quad \text{for} \quad \boldsymbol{Q}_{ij} = \sum_{a:i\to j} t_a + \sum_{b:j\to i} t_b \in \mathbb{K}\,.$$

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6. The invariant tensor of $(\cdot,\cdot)_{\mathfrak{g}}$ is the classical R-matrix $\boldsymbol{r}$, explicitly

$$\boldsymbol{r} = \boldsymbol{r}_{\mathrm{diag}} + \sum_{\alpha>0} \sum_s \left(e_\alpha^{(s)} \otimes e_{-\alpha}^{(s)} + (-1)^{|\alpha|} e_{-\alpha}^{(s)} \otimes e_\alpha^{(s)} \right),$$

$$\boldsymbol{r}_{\mathrm{diag}} = \sum_i (\mathbb{V}_i \otimes \mathbb{D}_i + \mathbb{D}_i \otimes \mathbb{V}_i) - \sum_{i,j} \boldsymbol{Q}_{ij} \mathbb{V}_i \otimes \mathbb{V}_j$$

where $e_\alpha^{(s)}$ form a basis in the root space $\mathfrak{g}_\alpha$ such that for $\alpha > 0$

$$(e_\beta^{(r)}, e_\alpha^{(s)}) = \delta_{\beta,-\alpha}\, \delta_{r,s}\,.$$

5. The restriction of $(\cdot,\cdot)_{\mathfrak{g}}$ to $\mathfrak{g}_0$ yields a non-degenerate symmetric bilinear form on $\mathfrak{g}_0 = \mathfrak{V} \oplus \mathfrak{D}$, which is explicitly given by the matrix

$$\begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & \boldsymbol{Q} \end{pmatrix} \quad \text{for} \quad \boldsymbol{Q}_{ij} = \sum_{a:i\to j} t_a + \sum_{b:j\to i} t_b \in \mathbb{K}\,.$$

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7. For $\alpha > 0$,

$$[e_\alpha^{(s)}, e_{-\alpha}^{(t)}] = (-1)^{|\alpha|}\, \delta_{s,t}\, h_\alpha \quad \text{where} \quad h_\alpha = \sum_i \alpha_i\, \mathbb{D}_i - \sum_{i,j} \boldsymbol{Q}_{ij}\, \alpha_i\, \mathbb{V}_j\,.$$

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Remark

For a tripled quiver $\widetilde{Q}$ with canonical cubic potential W_{can}, it follows from previous theorem that $\mathfrak{g}_{\widetilde{Q},W_{\mathrm{can}}} \cong \mathfrak{g}^{\mathrm{MO}}_{Q}$, the Maulik-Okounkov Lie algebra. It is proven in [Botta-Davison 2023] that $\mathfrak{g}^{\mathrm{MO},+}_{Q} \cong \mathfrak{g}^{\mathrm{BPS}}_{\widetilde{Q},W_{\mathrm{can}}}$, so the above conjecture is true in this case.

Relation to the double of CoHA

Let $\mathcal{SH}^{\mathrm{nilp}}_{Q,W}$ be the spherical nilpotent critical CoHA of (Q, W), with generators

$$\{e_{i,n}\}_{i\in Q_0,n\in\mathbb{Z}_{\geq 0}}, \quad e_{i,n}{=}c_1(\mathcal{L}_i^{\mathrm{taut}})^n{\cap}[\mathrm{Rep}_Q(\delta_i)^{\mathrm{nilp}}/\mathbb{C}^*]$$

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Definition

Define $\mathcal{DSH}_\infty(Q,W) :=$ free product $\mathcal{SH}^{\mathrm{nilp}}_{Q,W} * \mathcal{SH}^{\mathrm{nilp,op}}_{Q,W} * \mathbb{C}[h_{i,n} \mid i\in Q_0, n\in\mathbb{Z}]$ modulo relations

$$[e_{i,r},f_{j,s}]=(-1)^\sharp\delta_{ij}\gamma_i h_{i,r+s}, \text{ where } \gamma_i:=\textstyle\prod_{e:i\to i} t_e,$$

$$h_i(z)e_j(w)=\zeta_{ij}(z)e_j(w)h_i(z), \quad h_i(z)f_j(w)=\zeta_{ij}(z)^{-1}f_j(w)h_i(z),$$

$$\zeta_{ij}(z)=(-1)^\sharp\frac{\prod_{a:i\to j}(z-t_a-\sigma_j)}{\prod_{b:j\to i}(z+t_b-\sigma_j)}, \text{ where } \sigma_j \text{ is the operator } e_{j,r}\mapsto e_{j,r+1},\ f_{j,r}\mapsto f_{j,r+1}.$$

Proposition [COZZ 2025]

For an arbitrary quiver Q with potential W, and arbitrary framing $\underline{\mathbf{d}}$ with $\underline{\mathbf{d}}$-framed potential W^{fr}, there is a natural action

$$\mathcal{DSH}_\infty(Q,W) \curvearrowright \mathcal{H}_{\underline{\mathbf{d}}}^{W^{\mathrm{fr}}} = \bigoplus_{\mathbf{v}} H_{\mathrm{eq,crit}}(\mathcal{M}(\mathbf{v},\underline{\mathbf{d}}),W^{\mathrm{fr}})$$

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where $e_i(z)$ and $f_i(z)$ act by Hecke correspondences, and $h_i(z)$ acts by multiplication by Chern classes

$$(-1)^\sharp z^{\mathrm{rk}\,\mathcal{U}_i} c_{-1/z}(\mathcal{U}_i), \quad \mathcal{U}_i = \sum_{a:i\to j} t_a \mathsf{V}_j - \sum_{b:j\to i} t_b^{-1} \mathsf{V}_j + \mathsf{D}_{\mathrm{out},i} - \mathsf{D}_{\mathrm{in},i}$$

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Proposition [COZZ 2025]

Suppose that Q is symmetric, then the above action factors through μ-shifted double spherical CoHA

$$\mathcal{DSH}_\mu(Q,W) := \mathcal{DSH}_\infty(Q,W) \Big/ \Big(h_{i,t} = 0 \text{ for } t < -\mu_i - 1,\ h_{i,-\mu_i-1} = 1 \Big)$$

where $\mu = \mathbf{d}_{\mathrm{out}} - \mathbf{d}_{\mathrm{in}}$.

Suppose that Q is symmetric, then the action induces an algebra homomorphism

$$\mathcal{DSH}_\mu(Q,W) \to \mathrm{End}\left(\mathcal{H}^{W_1}_{\underline{\mathbf{d}_1}}(a_1) \otimes \cdots \otimes \mathcal{H}^{W_n}_{\underline{\mathbf{d}_n}}(a_n)\right) \text{ with } \mu = \sum_{i=1}^n (\mathbf{d}_{i,\mathrm{out}} - \mathbf{d}_{i,\mathrm{in}})$$

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Assume moreover that $\mu\leq 0$, then the image of the above map is contained in $\mathsf{Y}_\mu(Q,W)$.

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Assume moreover that $\mu \leq 0$, then the image of the above map is contained in $\mathrm{Y}_\mu(Q,W)$.

The proof uses an explicit computation of certain matrix elements of the R-matrix for $\mathcal{H}^{W}_{\underline{\delta_i}}\otimes\mathcal{H}^{W^{\mathrm{fr}}}_{\underline{\mathbf{d}}}$:

$$R(z)_{\text{certain }2\times 2\text{ block}}=\begin{pmatrix}1 & 0\\ \frac{(-1)^{|i|}t_i}{\gamma_i}f_i(z) & 1\end{pmatrix}\begin{pmatrix}g_i(z) & 0\\ 0 & g_i(z)h_i(z)\end{pmatrix}\begin{pmatrix}1 & e_i(z)\\ 0 & 1\end{pmatrix}$$

The CoHA-double construction gives maps

$$Y_\mu(\text{Lie algebra } \mathfrak{g}) \to \mathrm{Y}_\mu(\text{corresponding } Q,\ W)$$

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Examples:

- all symmetrizable Kac-Moody Lie algebras, including all classical types (ABCDEFG)
- $\mathfrak{gl}_{n|m}$ and $\widehat{\mathfrak{gl}}_{n|m}$
- exceptional Lie super algebra $D(2,1;\lambda)$
- and more to explore...

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- and more to explore...

Explicit computation of R-matrices in these examples seems to be a challenging problem.

Other aspects of this story:

1. $H_{\mathrm{eq,crit}}(X, W)$ is a module of equiv. quantum cohomology $QH_{\mathrm{eq}}(X)$,

$$c_1(\mathcal{L}) \star \cdot = c_1(\mathcal{L}) \cup \cdot - \sum_{\substack{\alpha>0 \\ \alpha\cdot\mu=0}} \frac{(\alpha, \mathcal{L})\, z^\alpha}{(-1)^{|\alpha|} - z^\alpha} \mathrm{Cas}_{\alpha,\mu}$$
$$- \sum_{\substack{\alpha>0 \\ \alpha\cdot\mu\neq 0}} (-1)^{|\alpha|}(\alpha, \mathcal{L})\, z^\alpha\, \mathrm{Cas}_{\alpha,\mu} + \text{scalar}\,,$$

2. There is also a K-theoretic version of critical stable envelopes, R-matrices, quantum loop groups, quantum critical K-theory, qKZ, etc.

Thank you!